Loop homology of moment-angle complexes in the flag case

(based on arXiv:2403.18450 and work in progress)

Fedor Vylegzhanin

Moscow State University / Higher School of Economics, Russia

Advances in Homotopy Theory VI
June 18, 2024 @ BIMSA

Outline

X a simply connected space, $\Bbbk$ a commutative ring with unit $\rightsquigarrow$ $H_*(\Omega X;\Bbbk)$ — a connected associative $\Bbbk$-algebra with unit (even a Hopf algebra if $H_*(\Omega X;\Bbbk)$ is free over $\Bbbk$).

Goal: give a presentation of $H_*(\Omega X;\Bbbk)$ by generators and relations.

We consider $X=\mathcal{Z}_\mathcal{K}$ and $X=EH\times_H\mathcal{Z}_\mathcal{K}$, where $\mathcal{K}$ is a flag simplicial complex, $\mathcal{Z}_\mathcal{K}$ is the moment-angle complex and $H\subset\mathbb{T}^m$ is a subtorus.

More generally, our approach applies for fibrations $X\to E\xrightarrow{p}B$ where Ωp has a homotopy section and algebras $H_*(\Omega E;\Bbbk)$, $H_*(\Omega B;\Bbbk)$ are known.

Moment-angle complexes and their partial quotients

Fix an abstract simplicial complex $\mathcal{K}$ on vertex set $[m] = \{1, \ldots, m\}$. The moment-angle complex is the following CW-complex:

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{J \in \mathcal{K}} \left(\prod_{i \in J} D^2 \times \prod_{i \in [m] \setminus J} S^1 \right) \subset (D^2)^m.$$

Clearly, $\mathbb{T}^m = (S^1)^m$ acts on $\mathcal{Z}_{\mathcal{K}}$. If a closed subgroup $H \subset \mathbb{T}^m$ acts freely on $\mathcal{Z}_{\mathcal{K}}$, the $\mathbb{T}^m/H$-space $\mathcal{Z}_{\mathcal{K}}/H$ is called a partial quotient of $\mathcal{Z}_{\mathcal{K}}$.
Up to an equivariant homeomorphism, this class contains all compact smooth toric varieties and quasitoric manifolds (Davis, Januszkiewicz).

We obtain $H \subset \mathbb{T}^m$ as $T_\lambda := \mathrm{Ker}(\lambda_* : \mathbb{T}^m \to \mathbb{T}^n)$ for some $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$.

Moment-angle complexes and their homotopy quotients

Fix an abstract simplicial complex $\mathcal{K}$ on vertex set $[m] = \{1, \ldots, m\}$. The moment-angle complex is the following CW-complex:

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{J \in \mathcal{K}} \left(\prod_{i \in J} D^2 \times \prod_{i \in [m] \setminus J} S^1 \right) \subset (D^2)^m.$$

Clearly, $\mathbb{T}^m = (S^1)^m$ acts on $\mathcal{Z}_{\mathcal{K}}$. For any closed subgroup $H \subset \mathbb{T}^m$ we call the $\mathbb{T}^m/H$-space $EH \times_H \mathcal{Z}_{\mathcal{K}}$ a homotopy quotient of $\mathcal{Z}_{\mathcal{K}}$.
Up to an equivariant homotopy equivalence, this class contains all smooth toric varieties and quasitoric manifolds (Davis, Januszkiewicz / Franz).

We obtain $H \subset \mathbb{T}^m$ as $T_\lambda := \mathrm{Ker}(\lambda_* : \mathbb{T}^m \to \mathbb{T}^n)$ for arbitrary $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$ of full rank, and denote $X(\mathcal{K}, \lambda) := ET_\lambda \times_{T_\lambda} \mathcal{Z}_{\mathcal{K}}$.

Some homotopy fibrations

Consider the Davis-Januszkiewicz space:

$$\mathrm{DJ}(\mathcal{K}) := \bigcup_{J\in\mathcal{K}} \left(\prod_{i\in J} \mathbb{C}\mathrm{P}^\infty \times \prod_{i\in[m]\setminus J} \mathrm{pt} \right) \subset (\mathbb{C}\mathrm{P}^\infty)^m.$$

Buchstaber, Panov: there are homotopy fibrations

$$\mathcal{Z}_\mathcal{K} \to \mathrm{DJ}(\mathcal{K}) \xrightarrow{p} (\mathbb{C}\mathrm{P}^\infty)^m,\ X(\mathcal{K},\lambda) \to \mathrm{DJ}(\mathcal{K}) \xrightarrow{p'} (\mathbb{C}\mathrm{P}^\infty)^n.$$

Panov, Ray: Ωp and $\Omega p'$ admit homotopy sections. Hence

$$\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_\mathcal{K} \times \mathbb{T}^m \simeq \Omega X(\mathcal{K},\lambda) \times \mathbb{T}^n;$$

in particular, $\pi_j(\mathrm{DJ}(\mathcal{K})) \simeq \pi_j(\mathcal{Z}_\mathcal{K}) \simeq \pi_j(X(\mathcal{K},\lambda))$ for $j \geq 2$.

Loop homology algebras

The results below use the split fibration $\Omega\mathcal{Z}_\mathcal{K} \to \Omega\mathrm{DJ}(\mathcal{K}) \to \mathbb{T}^m$ and (hga-)formality of $\mathrm{DJ}(\mathcal{K})$. Here $\Bbbk$ is arbitrary, $\Bbbk[\mathcal{K}]$ is the face ring of $\mathcal{K}$.

Theorem (Panov, Ray'08 / V.)

1. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)$ as graded $\Bbbk$-algebras;
2. $H_*(\Omega\mathrm{DJ}(\mathcal{K};\Bbbk) \cong H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\otimes\Lambda[u_1,\dots,u_m]$ as left $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$-modules;
3. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \hookleftarrow T(u_1,\dots,u_m)/(u_i^2=0,\ i=1,\dots,m;\ [u_i,u_j]=0,\ \{i,j\}\in\mathcal{K})$. This is the whole algebra if $\mathcal{K}$ is a flag simplicial complex (if $I\in\mathcal{K}$ whenever $\{i,j\}\in\mathcal{K}$ for all $i,j\in I$).

Theorem (Franz'21 / V.)

Suppose that $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ is a free $\Bbbk$-module (e.g. $\mathcal{K}$ is flag). Then

1. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)$ as Hopf $\Bbbk$-algebras;
2. $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\to H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)\to\Lambda[u_1,\dots,u_m]$ is an extension of Hopf algebras.

Stanton's homotopy decomposition

Theorem (Stanton'23)

Let $\mathcal{L}$ be a flag simplicial complex, and $\mathcal{K} = \mathcal{L}$ or $\mathcal{K} = \mathrm{sk}_d\,\mathcal{L}$. Then there is a homotopy equivalence

$$\Omega\mathcal{Z}_{\mathcal{K}} \simeq (S^3)^{\times B} \times (S^7)^{\times C} \times \prod_{\substack{n\geq 3,\\ n\neq 4,8}} (\Omega S^n)^{\times D_n}$$

for some $B, C, D_n \geq 0$. In particular,

$$\pi_N(\mathcal{Z}_{\mathcal{K}}) \simeq \pi_{N-1}(S^3)^{\oplus B} \oplus \pi_{N-1}(S^7)^{\oplus C} \oplus \bigoplus_{\substack{n\geq 3,\\ n\neq 4,8}} \pi_N(S^n)^{\oplus D_n}.$$

Our appoach: find B, C, D_n by computing the Poincaré series of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)$, using $\mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk) \cong H_*(\Omega \mathrm{DJ}(\mathcal{K});\Bbbk) \simeq H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)\otimes\Lambda[m]$.

Homotopy groups, the case $\mathcal{K} = \mathcal{K}^f$

Theorem (V.'24)

Let $\mathcal{K}$ be a flag simplicial complex on vertex set $[m]$. Then there is a homotopy equivalence

$$\Omega\mathcal{Z}_\mathcal{K} \simeq (S^3)^{\times B} \times (S^7)^{\times C} \times \prod_{\substack{n\geq 3,\\ n\neq 4,8}} (\Omega S^n)^{\times D_n},$$

where the numbers B, C, D_n satisfy

$$\frac{\prod_n (1-t^{n-1})^{D_n}}{(1+t^3)^B(1+t^7)^C} = (1+t)^{m-\dim(\mathcal{K})} h_\mathcal{K}(-t) = \sum_{J\subset[m]} (1-\chi(\mathcal{K}_J))\cdot t^{|J|}.$$

This allows to describe homotopy groups of smooth toric varieties and quasitoric manifolds corresponding to flag complexes!

Homotopy groups, the case $\mathcal{K} = \operatorname{sk}_d \mathcal{K}^f$ (work in progress)

Theorem (V., work in progress)

Let $\mathcal{L}$ be a flag simplicial complex and $\mathcal{K} = \operatorname{sk}_d \mathcal{L}$. Denote $M := \{I \in \mathcal{L} : |I| = d+2\}$ and $N := \{J \in \mathcal{L} : |J| = d+3\}$. Then

$$H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk) \cong T(u_1, \dots, u_m; w_I, I \in M)/\mathcal{I},$$

$$\mathcal{I} = \Big(u_i^2 = 0,\ i = 1, \dots, m;\ [u_i, u_j] = 0,\ \{i,j\} \in \mathcal{K};$$

$$([u_i, w_I] = 0,\ i \in I \in M;\ \sum_{i \in J} [u_i, w_{J \setminus i}] = 0,\ J \in N \Big).$$

Then the Poincaré series of $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk)$ can be computed using Gröbner bases...

Presentations of connected algebras ($\Bbbk$ is a field)

Let $\Bbbk$ be a commutative ring with unit. Graded associative $\Bbbk$-algebra $A = \bigoplus_{n\geq 0} A_n$ with unit is connected if $A_0 = \Bbbk \cdot 1$. A presentation of A is any isomorphism $A \simeq T(a_1, \dots, a_N)/(r_1 = \dots = r_M = 0)$, where a_i, r_j are homogeneous elements of positive degree.

Theorem (C.T.C.Wall'60)

Let $\Bbbk$ be a field and A be an associative graded $\Bbbk$-algebra. Let $n \geq 0$.

1. $\forall$ presentation of A has $\geq \dim_\Bbbk \mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n$ generators and $\geq \dim_\Bbbk \mathrm{Tor}_2^A(\Bbbk,\Bbbk)_n$ relations of degree n.
2. $\exists$ a presentation of A with $\dim_\Bbbk \mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n$ generators and $\dim_\Bbbk \mathrm{Tor}_2^A(\Bbbk,\Bbbk)_n$ relations of degree n.

Presentations of connected algebras ($\Bbbk$ is a PID)

Let $\Bbbk$ be a commutative ring with unit. Graded associative $\Bbbk$-algebra $A = \bigoplus_{n\geq 0} A_n$ with unit is connected if $A_0 = \Bbbk \cdot 1$. A presentation of A is any isomorphism $A \simeq T(a_1, \dots, a_N)/(r_1 = \dots = r_M = 0)$, where a_i, r_j are homogeneous elements of positive degree.

Theorem (V.)

Let $\Bbbk$ be a PID and A be an associative graded $\Bbbk$-algebra. Let $n \geq 0$.

1. $\forall$ presentation of A has $\geq \mathrm{gen}\,\mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n$ generators and $\geq \big(\mathrm{gen}\,\mathrm{Tor}_2^A(\Bbbk,\Bbbk)_n + \mathrm{rel}\,\mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n\big)$ relations of degree n.
2. $\exists$ a presentation of A with $\mathrm{gen}\,\mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n$ generators and $\big(\mathrm{gen}\,\mathrm{Tor}_2^A(\Bbbk,\Bbbk)_n + \mathrm{rel}\,\mathrm{Tor}_1^A(\Bbbk,\Bbbk)_n\big)$ relations of degree n.

Here $M \simeq \Bbbk^{\mathrm{gen}\,M}/\Bbbk^{\mathrm{rel}\,M}$, where $\mathrm{gen}\,M$ and $\mathrm{rel}\,M$ are the smallest possible. For example, $\mathrm{gen}(\mathbb{Z}/6 \oplus \mathbb{Z}) = 2$, $\mathrm{rel}(\mathbb{Z}/6 \oplus \mathbb{Z}) = 1$ if $\Bbbk = \mathbb{Z}$.

Presentations and the bar construction

The $\Bbbk$-module $\operatorname{Tor}^A(\Bbbk,\Bbbk)$ is isomorphic to homology of the bar construction $(\overline{\mathrm{B}}(A), d)$, where $\overline{\mathrm{B}}(A)_n = (A_{>0})^{\otimes n}$.

Theorem (V.)

Choose the elements

- $a_1,\dots,a_N \in A_{>0} \simeq \overline{\mathrm{B}}_1(A)$ whose images generate $\operatorname{Tor}_1^A(\Bbbk,\Bbbk)$;
- $\rho_r = \sum_\beta K_{r,\beta}\otimes L_{r,\beta} \in A_{>0}\otimes A_{>0} \simeq \overline{\mathrm{B}}_2(A)$ so that triviality of their images $d_{\overline{\mathrm{B}}}(\rho_r) \in \overline{\mathrm{B}}_1(A)$ give a sufficient set of additive relations between $[a_1],\dots,[a_N] \in \operatorname{Tor}_1^A(\Bbbk,\Bbbk)$;
- $\sum_\alpha P_{i,\alpha}\otimes Q_{i,\alpha} \in A_{>0}\otimes A_{>0} \simeq \overline{\mathrm{B}}_2(A)$ whose images generate $\operatorname{Tor}_2^A(\Bbbk,\Bbbk)$ as a $\Bbbk$-module.

Then we have a presentation

$$A \simeq T(a_1,\dots,a_N)/(\sum_\alpha \pm P_{i,\alpha}\cdot Q_{i,\alpha} = \sum_\beta \pm K_{r,\beta}\cdot L_{r,\beta} = 0).$$

Our approach to loop homology

Let $\mathcal{K}$ be a flag complex. We have: $S = H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is a subalgebra in the known algebra $A = H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$, and $A \simeq S \otimes_\Bbbk \Lambda[m]$ as a S-module. A presentation of S is computed as follows.

1. We have the Fröberg resolution $(A\otimes\Bbbk\langle\mathcal{K}\rangle, d)$ of the left A-module $\Bbbk$. Consider it as a free resolution $(S\otimes\Lambda[m]\otimes\Bbbk\langle\mathcal{K}\rangle,\widehat{d})$ of the left S-module $\Bbbk$.
2. Compute $\mathrm{Tor}^S(\Bbbk,\Bbbk)$ as homology of $(\Lambda[m]\otimes\Bbbk\langle\mathcal{K}\rangle,\overline{d})$.
3. Construct a homology isomorphism $\overline{\varphi} : (\Lambda[m]\otimes\Bbbk\langle\mathcal{K}\rangle,\overline{d}) \to (\overline{\mathrm{B}}(S), d)$.
4. Obtain elements in $\overline{\mathrm{B}}(S)$ corresponding to additive generators and relations in $\mathrm{Tor}^S(\Bbbk,\Bbbk)$.
5. Use the previous slide to give a presentation of S.

Theorem (V.'22)

On step 2 we obtain $\mathrm{Tor}_q^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk) \cong \bigoplus_{J\subset[m]} \widetilde{H}_{q-1}(\mathcal{K}_J;\Bbbk)$.

Results: generators for $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, flag case

Recall that the algebra $H_*(\Omega \mathrm{DJ}(\mathcal{K});\Bbbk)$ is generated by elements $u_1,\dots,u_m$ of degree 1. For $I=\{i_1<\dots<i_k\}\subset[m]$, $x\in H_*(\Omega \mathrm{DJ}(\mathcal{K});\Bbbk)$ denote

$$c(I,x) := [u_{i_1},[u_{i_2},\dots[u_{i_k},x]\dots]]\in H_*(\Omega \mathrm{DJ}(\mathcal{K});\Bbbk).$$

One can show that $c(I,u_j)\in H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\subset H_*(\Omega \mathrm{DJ}(\mathcal{K});\Bbbk)$ if $I\neq\varnothing$.

For every $J\subset[m]$ let $\Theta(J)\subset J$ contain exactly one vertex from every path component of $\mathcal{K}_J$ not containing $\max(J)$ (for example, the minimal vertices of path components). We have $|\Theta(J)|=b_0(\mathcal{K}_J)-1$.

Theorem (Grbić, Panov, Theriault, Wu'16 / V.)

$H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is multiplicatively generated by the GPTW generators $\{c(J\setminus j,u_j):\ J\subset[m], j\in\Theta(J)\}$. It is a minimal set of generators.

Results: relations in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, flag case

Each $c(I,u_j)$ can be expressed through the GPTW generators by an explicit recursive algorithm. Denote any such expression as $\widehat{c}(I,u_j)$.

Theorem (V.'24)

For every $J\subset[m]$ choose a set of simplicial 1-cycles $\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{\alpha}[\{i,j\}]$ in $C_1(\mathcal{K}_J;\Bbbk)$, whose images generate the $\Bbbk$-module $H_1(\mathcal{K}_J;\Bbbk)$. Then $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is presented by GPTW generators modulo the relations

$$\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus\{i,j\}=A\sqcup B:\\ \max(A)>i,\\ \max(B)>j.}}(-1)^{\cdots}\Big[\widehat{c}(A,u_i),\widehat{c}(B,u_j)\Big]=0.$$

In particular, there is a presentation by $\sum_{J\subset[m]}(b_0(\mathcal{K}_J)-1)$ generators and $\sum_{J\subset[m]}\operatorname{gen}H_1(\mathcal{K}_J;\Bbbk)$ relations. It is minimal among $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentations, if $\Bbbk$ is a PID.

An example: 5-cycle

For $\mathcal{K}$ a 5-cycle, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is presented by 10 generators and a single relation, first computed by Veryovkin (2016, computer bruteforce). The GPTW generators:

$$[u_3,u_1],\ [u_4,u_1],\ [u_4,u_2],\ [u_5,u_2],\ [u_5,u_3],\ [u_1,[u_5,u_3]],$$

$$[u_2,[u_4,u_1]],\ [u_3,[u_4,u_1]],\ [u_3,[u_5,u_2]],\ [u_4,[u_5,u_2]].$$

Our formula gives the relation:

$$-\Big[[u_3,u_1],[u_4,[u_5,u_2]]\Big]+\Big[[u_4,u_1],[u_3,[u_5,u_2]]\Big]+\Big[[u_3,[u_4,u_1]],[u_5,u_2]\Big]$$

$$+\Big[[u_4,u_2],[u_1,[u_5,u_2]]\Big]+\Big[\underline{[u_1,[u_4,u_2]]},[u_5,u_3]\Big]=0.$$

The underlined element is not a GPTW generator; however, $-[u_1,[u_4,u_2]]=[u_2,[u_4,u_1]]$ in $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$, and $[u_2,[u_4,u_1]]$ is a generator.

$\mathbb{Q}$-coformality in the flag case

Simply connected space X is $\Bbbk$-coformal if $C_*(\Omega X; \Bbbk) \sim H_*(\Omega X; \Bbbk)$ as dga algebras over $\Bbbk$. It is known that $\mathrm{DJ}(\mathcal{K})$ is coformal if and only if $\mathcal{K}$ is flag.

Proposition (V.'24)

If $\mathcal{K}$ is flag, then $\mathcal{Z}_\mathcal{K}$ and all $X(\mathcal{K}, \lambda)$ are $\mathbb{Q}$-coformal.

This follows from the following theorem.

Theorem (Huang'23)

Let $F \to E \to B$ be a fibration of nilpotent spaces of finite type, such that E is $\mathbb{Q}$-coformal and $\pi_*(F) \otimes \mathbb{Q} \to \pi_*(E) \otimes \mathbb{Q}$ is injective. Then F is $\mathbb{Q}$-coformal.

$\Bbbk$-coformality in the flag case?

X a simply connected space such that $H_*(\Omega X;\Bbbk)$ is $\Bbbk$-free $\rightsquigarrow$
Milnor-Moore spectral sequence $E^2_{p,q} = \mathrm{Tor}_p^{H_*(\Omega X;\Bbbk)}(\Bbbk,\Bbbk)_q \Rightarrow H_{p+q}(X;\Bbbk)$.
We have $E^2 = E^\infty$ if X is $\Bbbk$-coformal.

Theorem (V.'22)

If $\mathcal{K}$ is flag, then $E^2 = E^\infty$ for $\mathcal{Z}_\mathcal{K}$ for any $\Bbbk$.

Conjecture

If $\mathcal{K}$ is flag, then $\mathcal{Z}_\mathcal{K}$ and all $X(\mathcal{K},\lambda)$ are coformal over any $\Bbbk$.

It would follow from the following generalisation of Huang's result.

Conjecture

Let $F \to E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type, such that E is $\Bbbk$-coformal and Ωp has a homotopy section. Then F is $\Bbbk$-coformal.

Homotopy quotients, flag case

Similar approach to loop homology of $X(\mathcal{K},\lambda)=ET^n\times_{T^n}\mathcal{Z}_\mathcal{K}$ gives

$$\mathrm{Tor}^{H_*(\Omega X(\mathcal{K},\lambda);\Bbbk)}(\Bbbk,\Bbbk)\simeq H[\Lambda[t_1,\dots,t_n]\otimes\Bbbk\langle\mathcal{K}\rangle,d]\simeq H_*(X(\mathcal{K},\lambda);\Bbbk).$$

In general, we do not know the homology of this complex!

Theorem (V., work in progress)

Suppose that $\mathcal{K}$ is flag and $X(\mathcal{K},\lambda)$ is a quasitoric manifold (or a partial quotient $\mathcal{Z}_\mathcal{K}/T^n$, where $\mathcal{K}$ is a Cohen-Macaulay complex of dimension $n-1$). Then $H_*(\Omega X(\mathcal{K},\lambda);\Bbbk)$ is presented by $h_1(\mathcal{K})=m-n$ generators of degree 1 (linear combinations of $u_1,\dots,u_m$) modulo $h_2(\mathcal{K})$ relations of degree 2.

Results of Berglund on Koszul spaces imply that this algebra is quadratic dual to $H^*(X(\mathcal{K},\lambda);\Bbbk)\cong\Bbbk[\mathcal{K}]/(t_1,\dots,t_n)$ if $\Bbbk=\mathbb{Q}$, and both these algebras are Koszul. I do not know how to prove it algebraically.

Open questions

1. Suppose that $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk)$ has additive torsion (examples are known). Is the comultiplication well defined?
2. For which complexes $\mathcal{K}$ the Hopf algebra $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk)$ is primitively generated? (Always the case if $\Bbbk$ is a field.)
3. Give an explicit dga model for $C_*(\Omega \mathcal{Z}_\mathcal{K}; \Bbbk)$. In particular: prove (or disprove) that $C_*(\Omega \mathcal{Z}_\mathcal{K}; \Bbbk) \sim \Omega[C_*^{\mathrm{CW}}(\mathcal{Z}_\mathcal{K}; \Bbbk)]$.
4. Are moment-angle complexes $\mathbb{Z}$-coformal in the flag case?
5. Describe Whitehead products in $\pi_*(\mathcal{Z}_\mathcal{K})$ whenever Stanton's decomposition $\Omega \mathcal{Z}_\mathcal{K} \simeq \prod_\alpha S^\alpha \times \prod_\beta \Omega S^\beta$ holds.
6. Is the Stanton's decomposition "$\mathbb{Z}^m_{\geq 0}$-graded"?
7. Compute $H^*(X(\mathcal{K}, \lambda); \Bbbk) \simeq H[\Lambda[n] \otimes \Bbbk[\mathcal{K}], d]$ (at least additively).
8. In the case of quasitoric manifolds for flag complexes: prove that $H^*(X; \Bbbk)$ and $H_*(\Omega X; \Bbbk)$ are quadratic dual.

Bibliography (1/3)

- C. T. C. Wall. Generators and relations for the Steenrod algebra.
  Ann. of Math. 72 (1960), no. 3, 429-444.
- R. Fröberg. Determination of a class of Poincaré series.
  Math. Scand. 37 (1975), no. 1, 29-39.
- M. W. Davis and T. Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions.
  Duke Math. J. 62 (1991), 417-451.
- T. E. Panov and N. Ray. Categorical aspects of toric topology.
  In: *Toric Topology*, M. Harada et al., eds. Contemp. Math., vol. 460. Amer. Math. Soc., Providence, RI, 2008, pp. 293–322.
- A. Berglund. Koszul spaces.
  Trans. Amer. Math. Soc. 366(9) (2014), 4551-4569.

Bibliography (2/3)

- V. M. Buchstaber and T. E. Panov. *Toric topology*, volume 204 of *Mathematical Surveys and Monographs*.
  AMS, Providence, RI (2015).

- J. Grbić, T. Panov, S. Theriault and J. Wu. The homotopy types of moment-angle complexes for flag complexes.
  Trans. Amer. Math. Soc. 368 (2016), 6663-6682.

- Ya. A. Verevkin. Pontryagin algebras of some moment-angle-complexes.
  Dal'nevost. Mat. Zh., 16:1 (2016), 9-23.

- M. Franz. Homotopy Gerstenhaber formality of Davis-Januszkiewicz spaces.
  Homology Homotopy Appl. 23 (2021), 325-347.

Bibliography (3/3)

F. E. Vylegzhanin. Pontryagin algebras and the LS-category of moment-angle complexes in the flag case.
Proc. Steklov Inst. Math. 317 (2022), 55-77.

R. Huang. Coformality around fibrations and cofibrations.
Homology Homotopy Appl. 25 (2023), 1, 235-248.

L. Stanton. Loop space decompositions of moment-angle complexes associated to flag complexes.
Quart. J. Math (haae020, 2024). arXiv:2306.12814.

F. Vylegzhanin. Loop homology of moment-angle complexes in the flag case.
Preprint (2024), arXiv:2403.18450.

Thank you for your attention!

Thank you for your attention!

Number of generators and relations in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$

There is a natural $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$. Let $\Bbbk$ be a PID and $\mathcal{K}$ be a flag complex on $[m]$.

Theorem (V.'24)

1. $\exists$ a $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ by $\sum_{J\subset[m]}(b_0(\mathcal{K}_J)-1)$ generators modulo $\sum_{J\subset[m]}\operatorname{gen} H_1(\mathcal{K}_J;\Bbbk)$ relations.
2. $\forall$ $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ has $\geq(b_0(\mathcal{K}_J)-1)$ generators and $\geq\operatorname{gen} H_1(\mathcal{K}_J;\Bbbk)$ relations of degree $(-|J|,2J)$.

This follows from $\operatorname{Tor}_p^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)\simeq\bigoplus_{J\subset[m]}\widetilde{H}_{p-1}(\mathcal{K}_J;\Bbbk)$.

Number of generators and relations in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$

There is a natural $\mathbb{Z}$-grading on $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$. Let $\Bbbk$ be a PID and $\mathcal{K}$ be a flag complex on $[m]$.

Theorem (V.'24)

1. $\exists$ a $\mathbb{Z}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ by $\sum_{J\subset[m]}(b_0(\mathcal{K}_J)-1)$ generators modulo $\sum_{n\geq 0}\operatorname{gen}\left(\bigoplus_{|J|=n} H_1(\mathcal{K}_J;\Bbbk)\right)$ relations.
2. $\forall$ $\mathbb{Z}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ has $\geq \sum_{|J|=n}(b_0(\mathcal{K}_J)-1)$ generators and $\geq \operatorname{gen}\left(\bigoplus_{|J|=n} H_1(\mathcal{K}_J;\Bbbk)\right)$ relations of degree n.

This follows from $\operatorname{Tor}_p^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)\simeq\bigoplus_{J\subset[m]}\widetilde{H}_{p-1}(\mathcal{K}_J;\Bbbk)$.