Equivariant cohomology of moment-angle complexes with respect to coordinate subtori

joint work with Indira Zeinikesheva

Taras Panov

Moscow State University

Algebraic Groups: the White Nights Season II

4–8 July 2022

The polyhedral product functor

$(\mathbf{X},\mathbf{A}) = \{(X_1,A_1),\ldots,(X_m,A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1,2,\ldots,m\}$, $\varnothing \in \mathcal{K}$.

Given $I = \{i_1,\ldots,i_k\} \subset [m]$, set

$$(\mathbf{X},\mathbf{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I,\\ A_i & \text{if } i \notin I.\end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\mathbf{X},\mathbf{A})$ is

$$(\mathbf{X},\mathbf{A})^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (\mathbf{X},\mathbf{A})^I = \bigcup_{I\in\mathcal{K}} \Bigl(\prod_{i\in I} X_i \times \prod_{i\notin I} A_i\Bigr) \subset \prod_{i=1}^m X_i.$$

Notation: $(X,A)^{\mathcal{K}} = (\mathbf{X},\mathbf{A})^{\mathcal{K}}$ when all $(X_i,A_i) = (X,A)$;
$\mathbf{X}^{\mathcal{K}} = (\mathbf{X},pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X,pt)^{\mathcal{K}}$.

Categorical approach

Category of faces $\mathrm{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\mathbf{X},\mathbf{A})\colon \mathrm{CAT}(\mathcal{K}) &\longrightarrow \mathrm{TOP}, \\ I &\longmapsto (\mathbf{X},\mathbf{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\mathrm{CAT}(\mathcal{K})$ to the inclusion of spaces $(\mathbf{X},\mathbf{A})^I \subset (\mathbf{X},\mathbf{A})^J$.

Then we have

$$(\mathbf{X},\mathbf{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\mathbf{X},\mathbf{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\mathbf{X},\mathbf{A})^I.$$

The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is the polyhedral product $(D^2, S^1)^{\mathcal{K}}$

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1 \Bigr) \subset (D^2)^m.$$

$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus T^m.
When $\mathcal{K}$ is a simplicial subdivision of sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

Also, consider the polyhedral product

$$U(\mathcal{K}) := (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_\geqslant \langle \mathbf{e}_i \colon i \in I \rangle \colon I \in \mathcal{K}\},$$

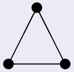
where $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\} \notin \mathcal{K}} \{z_{i_1} = \cdots = z_{i_k} = 0\}$

(the complement to a coordinate subspace arrangement);

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Equivariant cohomology

G a topological group, X a G-space. The Borel construction is

$$EG \times_G X := EG \times X/(e \cdot g^{-1}, g \cdot x) \sim (e, x),$$

where EG is the universal right G-space, $e \in EG$, $g \in G$, $x \in X$.
There is the Borel fibration $EG \times_G X \to BG$ over the classifying space $BG = EG/G$ with fibre X.

Equivariant cohomology of X is

$$H^*_G(X) := H^*(EG \times_G X).$$

The torus $T^m = (S^1)^m$ acts on $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ coordinatewise. The universal bundle $ES^1 \to BS^1$ is the infinite-dimensional Hopf bundle $S^\infty \to \mathbb{C}P^\infty$.

It is well known that the T^m-equivariant cohomology ring of $\mathcal{Z}_\mathcal{K}$ (or $U(\mathcal{K})$) is isomorphic to the face ring of $\mathcal{K}$ (the Stanley–Reisner ring):

$$H^*_{T^m}(\mathcal{Z}_\mathcal{K}) \cong \mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_\mathcal{K},$$

where $\mathcal{I}_\mathcal{K}$ is the ideal generated by the square-free monomials $v_I = \prod_{i \in I} v_i$ for which $I \subset [m]$ is not a simplex of $\mathcal{K}$.

For the ordinary cohomology ring of $\mathcal{Z}_\mathcal{K}$, we have

Theorem

There are isomorphisms of graded rings

$$\begin{aligned} H^*(\mathcal{Z}_\mathcal{K}) &\cong \mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\bigl(\mathbb{Z}, \mathbb{Z}[\mathcal{K}]\bigr) \\ &\cong H^*\bigl(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\bigr), \quad du_i = v_i,\ dv_i = 0, \end{aligned}$$

where $\Lambda[u_1, \ldots, u_m]$ *is the exterior algebra,* $\deg u_i = 1$, $\deg v_i = 2$.

We consider equivariant cohomology of $\mathcal{Z}_\mathcal{K}$ with respect to the action of coordinate subtori

$$T_I = \{(t_1, \ldots, t_m) \in T^m \colon t_j = 1 \text{ for } j \notin I\},$$

where $I = \{i_1, \ldots, i_k\} \subset [m]$.

Remark

$H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ is also the equivariant cohomology ring of $U(\mathcal{K})$ with respect to the coordinate subtorus $(\mathbb{C}^\times)_I \subset (\mathbb{C}^\times)^m$. It is also the same as the equivariant cohomology of the quotient toric variety $V_\Sigma = U(\mathcal{K})/G_\Sigma$ under the action of the subtorus given by the composite $(\mathbb{C}^\times)_I \to (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^m/G_\Sigma$.

We introduce two commutative dga models for the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$.

First, consider the dga

$$\big(\Lambda[u_i\colon i\notin I]\otimes\mathbb{Z}[\mathcal{K}],d\big),\quad du_i=v_i,\ dv_i=0,$$

where $\Lambda[u_i\colon i\notin I]$ is the exterior algebra on generators indexed by $V-I$. The grading is given by $\deg u_i=1$, $\deg v_i=2$.

Second, consider the quotient dga

$$R_I(\mathcal{K}):=\Lambda[u_i\colon i\notin I]\otimes\mathbb{Z}[\mathcal{K}]\big/(u_iv_i=v_i^2=0,\ i\notin I),$$

noting that the ideal generated by u_iv_i and v_i^2 with $i\notin I$ is d-invariant.

Theorem

The singular cochain algebra $C^*(ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}})$ *is quasi-isomorphic to* $(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d)$ *and* $R_I(\mathcal{K})$. *The quasi-isomorphisms are natural with respect to inclusion of subcomplexes.*

Proof

There is a polyhedral product decomposition (up to homotopy)

$$ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}} \xrightarrow{\simeq} (\mathbf{Y}, \mathbf{B})^{\mathcal{K}},$$

where

$$Y_i = \begin{cases} \mathbb{C}P^\infty, & i \in I, \\ D^2, & i \notin I, \end{cases} \qquad B_i = \begin{cases} pt, & i \in I, \\ S^1, & i \notin I. \end{cases}$$

The polyhedral product $(\mathbf{Y}, \mathbf{B})^{\mathcal{K}}$ interpolates between $(D^2, S^1)^{\mathcal{K}}$ (for $I = \varnothing$) and $(\mathbb{C}P^\infty, pt)^{\mathcal{K}}$ (for $I = [m]$).

Proof (continued)

First consider the case $\mathcal{K} = \Delta^{m-1} = \Delta[m]$, the full simplex on $[m]$. From the Eilenberg–Zilber and the Künneth theorems we obtain a zig-zag of quasi-isomorphisms

$$\begin{aligned} R_I(\Delta[m]) &= \Lambda[u_i\colon i\notin I]\otimes\mathbb{Z}[v_1,\ldots,v_m]\big/(u_iv_i, v_i^2\colon i\notin I)\\ &= \bigotimes_{i\in I}\mathbb{Z}[v_i]\otimes\bigotimes_{i\notin I}\bigl(\Lambda[u_i]\otimes\mathbb{Z}[v_i]/(u_iv_i,v_i^2)\bigr)\xrightarrow{\simeq}\bigotimes_{i\in I}C^*(\mathbb{C}P^\infty)\otimes\bigotimes_{i\notin I}C^*(D^2)\\ &\xrightarrow{\simeq}\cdots\xleftarrow{\simeq}C^*\Bigl(\prod_{i\in I}\mathbb{C}P^\infty\times\prod_{i\notin I}D^2\Bigr) = C^*\bigl((\mathbf{Y},\mathbf{B})^{\Delta[m]}\bigr), \end{aligned}$$

which completes the proof for the case $\mathcal{K} = \Delta[m]$.

Proof (continued)

Given a subset $J \subset [m]$, let $\Delta(J)$ denote a simplex on J, viewed as a simplicial complex on $[m]$ (with ghost vertices $[m] - J$). Then

$$R_I(\Delta(J)) = \Lambda[u_i \colon i \notin I] \otimes \mathbb{Z}[v_j \colon j \in J] \big/ (u_j v_j = v_j^2 = 0,\ j \in J - I),$$

Consider the $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$-diagram

$$\mathcal{R}_{I,\mathcal{K}} \colon \mathrm{CAT}^{\mathrm{op}}(\mathcal{K}) \longrightarrow \mathrm{DGA}, \quad J \longmapsto R_I(\Delta(J)),$$

sending a morphism $J_1 \subset J_2$ of $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$ to the surjection of dgas $R_I(\Delta(J_2)) \to R_I(\Delta(J_1))$. Then

$$R_I(\mathcal{K}) = \lim \mathcal{R}_{I,\mathcal{K}} = \lim\nolimits_{J \in \mathcal{K}} R_I(\Delta(J))$$

Proof (continued).

Similarly, we have a $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$-diagram

$$\mathcal{C}_{I,\mathcal{K}} \colon \mathrm{CAT}^{\mathrm{op}}(\mathcal{K}) \longrightarrow \mathrm{DGA}, \quad J \longmapsto C^*((\mathbf{Y},\mathbf{B})^J).$$

The quasi-isomorphisms for the case $\mathcal{K} = \Delta[m]$ imply an objectwise weak equivalence of diagrams $\mathcal{R}_{I,\mathcal{K}} \simeq \mathcal{C}_{I,\mathcal{K}}$. Furthermore, both diagrams are Reedy fibrant, so their limits are quasi-isomorphic. Thus, we obtain the required zig-zag of quasi-isomorphisms

$$\begin{aligned} R_I(\mathcal{K}) = \lim\nolimits_{J\in\mathcal{K}} R_I(\Delta(J)) \simeq \lim\nolimits_{J\in\mathcal{K}} C^*((\mathbf{Y},\mathbf{B})^J) \xleftarrow{\simeq} C^*(\operatorname*{colim}_{J\in\mathcal{K}}(\mathbf{Y},\mathbf{B})^J) \\ = C^*((\mathbf{Y},\mathbf{B})^{\mathcal{K}}) \simeq C^*(ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}}), \end{aligned}$$

where the last map in the top line is a quasi-isomorphism by excision (or by Mayer–Vietoris). □

For equivariant cohomology, we obtain

Theorem

There are isomorphisms of rings

$$H^*_{T_I}(\mathcal{Z}_\mathcal{K}) \cong H^*\bigl(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d\bigr) \cong H^*\bigl(R_I(\mathcal{K}), d\bigr)$$
$$\cong \mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\bigl(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}]\bigr),$$

where $\mathbb{Z}[v_i\colon i \in I]$ *is the* $\mathbb{Z}[v_1,\ldots,v_m]$*-module via the homomorphism sending* v_i *to* 0 *for* $i \notin I$*.*

When $I = [m]$, we obtain that the singular cochain algebra of $ET^m \times_{T^m} \mathcal{Z}_\mathcal{K} \simeq (\mathbb{C}P^\infty, pt)^\mathcal{K}$ is quasi-isomorphic to $\mathbb{Z}[\mathcal{K}]$ with zero differential, which is the integral formality result of Notbohm and Ray.

When $I = \varnothing$, we recover the description of the ordinary integral cohomology of $\mathcal{Z}_\mathcal{K}$.

Equivariant formality

A T^k-space X is called equivariantly formal if $H^*_{T^k}(X)$ is free as a module over $H^*_{T^k}(pt) = H^*(BT^k)$. The latter condition implies that the spectral sequence of the bundle $ET^k \times_{T^k} X \to BT^k$ collapses at the E_2 page.

By the previous calculation, $\mathcal{Z}_\mathcal{K}$ is equivariantly formal with respect to the action of T_I if $\mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}])$ is free as a module over $H^*(BT_I) = \mathbb{Z}[v_i\colon i \in I]$.

The join of simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ on the sets V_1 and V_2 is

$$\mathcal{K}_1 * \mathcal{K}_2 = \{I_1 \sqcup I_2 \subset V_1 \sqcup V_2 \colon I_1 \in \mathcal{K}_1,\ I_2 \in \mathcal{K}_2\}.$$

Theorem

Let $\mathcal{K}$ be a simplicial complex on a finite set V. The following conditions are equivalent:

(a) *For any $I \in \mathcal{K}$, the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ is a free module over $H^*(BT_I)$.*

(b) *There is a partition $V = V_1 \sqcup \cdots \sqcup V_p \sqcup U$ such that*

$$\mathcal{K} = \partial\Delta(V_1) * \cdots * \partial\Delta(V_p) * \Delta(U),$$

where $\Delta(U)$ denotes a full simplex on U, and $\partial\Delta(V_i)$ denotes the boundary of a simplex on V_i.

(c) *The rational face ring $\mathbb{Q}[\mathcal{K}]$ is a complete intersection ring (the quotient of the polynomial ring by an ideal generated by a regular sequence).*

Proof of (a)⇒(b)

We have $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \ldots, v_m]/(t_1, \ldots, t_p)$ where $t_k = \prod_{i \in V_k} v_i$ is a square-free monomial and V_k is a missing face of $\mathcal{K}$, for $k = 1, \ldots, p$. Suppose some of these missing faces intersect nontrivially, say, $V_1 \cap V_2 \neq \varnothing$. Then $I = V_1 - V_2$ is nonempty, and one can see that $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ is not a free $H^*(BT_I)$-module. A contradiction. Hence, $V_1, \ldots, V_p$ are pairwise non-intersecting, so $\mathcal{K}$ is as described in (b).

Proof of (b)⇒(a)

Write $I = I_1 \sqcup \cdots \sqcup I_p \sqcup J$, where $I_k \subsetneq V_k$, $J \subset U$. Then $\mathcal{Z}_\mathcal{K} = \mathcal{Z}_{\Delta(V_1)} \times \cdots \mathcal{Z}_{\partial\Delta(V_p)} \times \mathcal{Z}_{\Delta(U)}$ and $H^*_{T_{I_k}}(\mathcal{Z}_{\Delta(V_k)})$ is a free $H^*(BT_{I_k})$-module.

Proof of (b)⇒(c)

A sequence of homogeneous elements $(t_1, \dots, t_k)$ of positive degree in $\mathbb{Q}[v_1, \dots, v_m]$ is a regular sequence if t_{i+1} is not a zero divisor in $\mathbb{Q}[v_1, \dots, v_m]/(t_1, \dots, t_i)$ for $0 \leqslant i < k$. If $\mathcal{K}$ is as in (b), then $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \dots, v_m]/(t_1, \dots, t_p)$, where $m = |V|$ and $t_k = \prod_{i \in V_k} v_i$ for $k = 1, \dots, p$. Then $(t_1, \dots, t_p)$ is a regular sequence, so $\mathbb{Q}[\mathcal{K}]$ is a complete intersection ring.

Proof of (c)⇒(b)

Suppose $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1, \dots, v_m]/(t_1, \dots, t_p)$ where $(t_1, \dots, t_p)$ is a regular sequence. We can assume that $t_k = \prod_{i \in V_k} v_i$ where V_k is a missing face of $\mathcal{K}$, for $k = 1, \dots, p$. Suppose some of these missing faces intersect nontrivially, say, $V_1 \cap V_2 \neq \varnothing$. Then $t_2 \cdot \prod_{i \in V_1 - V_2} v_i = t_1 \cdot \prod_{j \in V_2 - V_1} v_j$, so t_2 is a zero divisor in $\mathbb{Q}[v_1, \dots, v_m]/(t_1)$. A contradiction. Hence, $V_1, \dots, V_p$ are pairwise non-intersecting, so $\mathcal{K}$ is as described in (b).

References

[1] Taras Panov and Indira Zeinikesheva. *Equivariant cohomology of moment-angle complexes with respect to coordinate subtori.* arXiv:2205.14678.