Quantum-classical duality between the Ruijsenaars-Schneider model and the quantum twisted spin chain: the trigonometric/anisotropic case

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based on work with A. Liashyk, A. Zabrodin, A. Zotov

arxiv:1510.07509; Nuclear Physics B, 903 (2016) 150-163

International Conference on Integrable Systems and Quantum symmetries (ISQS-24), Prague, 2016

Motivation

Heisenberg XXX chain on N sites - classical N-body Ruijsenaars-Schneider

A. Gorsky, A. Zabrodin and A. Zotov, *Spectrum of quantum transfer matrices via classical many-body systems*, JHEP 01 (2014) 070, [arXiv:1310.6958].

Classical Toda chain:

A. Givental and B.-S. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Commun. Math. Phys. 168 (1995) 609-641 [arXiv:hep-th/9312096]

Quantum Gaudin - classical Calogero-Moser:

E. Mukhin, V. Tarasov and A. Varchenko, *Gaudin Hamiltonians generate the Bethe alge- bra of a tensor power of vector representation of gl(N)*, St. Petersburg Math. J. 22 (2011) 463-472 [arXiv:0904.2131]

E. Mukhin, V. Tarasov and A. Varchenko, *Bethe algebra of Gaudin model, Calogero- Moser space and Cherednik algebra*, Int. Math. Res. Not. 2014 (2014) Issue 5 1174-1204 [arXiv:0906.5185]

Motivation

Spin chain Hamiltonians - modified Kadomtsev-Petviashvili:

A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi and A. Zabrodin, *Classical tau-function for quantum spin chains*, JHEP 1309 (2013) 064 [arXiv:1112.3310]

A. Zabrodin, *The master T-operator for inhomogeneous XXX spin chain and mKP hierarchy* SIGMA 10 (2014) 006 (18 pages), [arXiv:1310.6988]

A. Zabrodin, *The master T-operator for vertex models with trigonometric R-matrices as classical tau-function*, Teor. Mat. Fys. 171:1 (2013) 59-76 (Theor. Math. Phys. 174 (2013) 52-67) [arXiv:1205.4152]

A. Zabrodin, *Hirota equation and Bethe ansatz in integrable models, Suuri-kagaku Journal* (in Japanese), Number 596 (2013) 7-12

Statement

$$L_{ij}^{\mathrm{RS}} = \frac{\nu\,\dot{q}_j}{q_i - q_j + \eta\nu}$$

$$\hat{T}^{\mathrm{XXX}}(z) = \mathrm{tr}\,V + \sum_{j=1}^{N} \frac{\hat{H}_j^{\mathrm{XXX}}}{z - z_j}$$

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Substitute

$$\nu\eta = \hbar,\ q_j = z_j$$

$$\dot{q}_j = \frac{\eta}{\hbar}\, H_j^{\mathrm{XXX}}\left(\{q_i\}_N; \{\mu_i^1\}_{N_1}, \ldots, \{\mu_i^{n-1}\}_{N_{n-1}}\right)$$

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Now the spectrum reads

$$\big(\underbrace{V_1, \ldots, V_1}_{N-N_1}, \underbrace{V_2, \ldots, V_2}_{N_1-N_2}, \ldots, \underbrace{V_{n-1}, \ldots, V_{n-1}}_{N_{n-2}-N_{n-1}}, \underbrace{V_n, \ldots, V_n}_{N_{n-1}}\big)$$

Statement

$$L^{\rm RS}_{ij} = \frac{\nu\, \dot q_j}{q_i - q_j + \eta\nu} \qquad \hat T^{\rm XXX}(z) = {\rm tr}\, V + \sum_{j=1}^N \frac{\hat H^{\rm XXX}_j}{z - z_j}$$

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Get from XXX to XXZ, from rational to trigonometric:

$$\Big\{\underbrace{e^{-(N-N_1-1)\hbar}\, V_1, \ldots, e^{(N-N_1-1)\hbar}\, V_1}_{N-N_1}, \underbrace{e^{-(N_1-N_2-1)\hbar}\, V_2, \ldots, e^{(N_1-N_2-1)\hbar}\, V_2}_{N_1-N_2}, \ldots,$$

$$\ldots, \underbrace{e^{-(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}, \ldots, e^{(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}}_{N_{n-2}-N_{n-1}}, \underbrace{e^{-(N_{n-1}-1)\hbar}\, V_n, \ldots, e^{(N_{n-1}-1)\hbar}\, V_n}_{N_{n-1}}\Big\}$$

Heisenberg spin chain

$$S_j^\alpha = \frac{\hbar}{2}\sigma_j^\alpha \quad \text{live in} \quad \mathcal{H} = \bigotimes_{j=1}^N \mathcal{H}_j$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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so

$$S_j^\alpha = 1^{\otimes(j-1)} \otimes S^\alpha \otimes 1^{\otimes(N-j)}$$

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Most general case

$$\hat{H}_{XYZ} = \sum_{j=1}^N \left[J^x_{j(j+1)} S^x_j S^x_{j+1} + J^y_{j(j+1)} S^y_j S^y_{j+1} + J^z_{j(j+1)} S^y_j S^y_{j+1}\right]$$

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$J^\alpha_{j(j+1)} \equiv J^\alpha$ - homogeneity

$J^x = J^y = J^z$ - XXX case

$J^x = J^y \neq J^z$ - XXZ case

Heisenberg spin chain

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can rewrite in

$$S^{\pm}_j = S^x_j \pm i S^y_j \qquad \left[S^z_j, S^{\pm}_k \right] = \pm \delta_{jk} S^{\pm}_j, \quad \left[S^+_j, S^-_k \right] = 2\delta_{jk} S^z_j$$

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homogeneous XXZ then reads

$$H_{XXZ} = J \sum_{j=1}^{N} \left[\frac{1}{2} \left(S^+_j S^-_{j+1} + S^-_j S^+_{j+1} \right) + \Delta \left(S^z_j S^z_{j+1} - \frac{1}{4} \right) \right]$$

Transfer matrix

the integrals of motion $[\hat{H}, \mathcal{O}_i] = 0, \quad [\mathcal{O}_i, \mathcal{O}_j] = 0, \quad \forall i, j$

are generated by $T(\lambda) = \exp\left(\sum_{k=0}^{\infty} \frac{c_k}{k!} \mathcal{O}_k (\lambda - \lambda_0)^k \right)$

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so that $\mathcal{O}_k = 1/c_k \frac{d^k}{d\lambda^k} \log T(\lambda)|_{\lambda=\lambda_0}$

$[T(\lambda), T(\mu)] = 0, \quad \lambda \neq \mu$ holds true

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Introduce monodromy $\mathcal{T}(\lambda)$ - operator on $\mathcal{H}_0 \otimes \mathcal{H}$

then

$$T(\lambda) = \mathrm{Tr}_{\mathcal{H}_0} \mathcal{T}(\lambda)$$

Get to Yang-Baxter

introducing

$$\mathcal{T}_1(\lambda) = \mathcal{T}(\lambda) \otimes \hat{1}, \quad \mathcal{T}_2(\lambda) = \hat{1} \otimes \mathcal{T}(\lambda)$$

acting on $\quad \mathcal{H}_0^{(1)} \otimes \mathcal{H}_0^{(2)} \otimes \mathcal{H}$

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then from the commutation and $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}A \cdot \mathrm{Tr}B$

$$\mathrm{Tr}_{\mathcal{H}_0^{(1)} \otimes \mathcal{H}_0^{(2)}} [\mathcal{T}_1(\lambda), \mathcal{T}_2(\mu)] = 0$$

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so (tr. inv. under cycl. shift) exists an operator on $\mathcal{H}_0^{(1)} \otimes \mathcal{H}_0^{(2)}$

$$R_{12}(\lambda,\mu)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu)R_{12}^{-1}(\lambda,\mu) = \mathcal{T}_2(\lambda)\mathcal{T}_1(\mu)$$

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from here one gets to YB

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)$$

R-matrix

$$R(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

elements - eigenvalues of blocks $\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

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elements - eigenvalues of blocks $\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

$$b(\lambda) = \frac{\lambda}{\lambda+\eta}, \quad c(\lambda) = \frac{\eta}{\lambda+\eta} \text{ – XXX, case } \; |\Delta| = 1$$

$$b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda+\eta)}, \quad c(\lambda) = \frac{\sinh(\eta)}{\sinh(\lambda+\eta)} \text{ – XXZ, } \; |\Delta| < 1$$

with $\Delta = \cosh(\eta)$

Ruijsenaars-Schneider model

$\dot{q}_i = \partial_{p_i}\mathcal{H}, \quad \dot{p}_i = -\partial_{q_i}\mathcal{H}$ rewritten as $\dot{\mathcal{L}} = [M, \mathcal{L}]$,

$$\mathcal{H} = \eta^{-1}\sum_{i=1}^{N} e^{-\eta p_i}\prod_{k\neq i}^{N}\frac{q_i - q_k + \eta}{q_i - q_k}$$

$$\mathcal{H} = \sum_{i=1}^{N} e^{\eta p_i}\prod_{k\neq i}^{N}\frac{\sinh(q_i - q_k - \eta)}{\sinh(q_i - q_k)}$$

Ruijsenaars-Schneider model

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$$\mathcal{H} = \sum_{i=1}^{N} e^{\eta p_i}\prod_{k\neq i}^{N}\frac{\sinh(q_i - q_k - \eta)}{\sinh(q_i - q_k)}$$

$$\mathcal{H} = \mathrm{Tr}\,\mathcal{L}, \quad \mathcal{H}_k = \mathrm{Tr}(\mathcal{L}^k)$$

$$\mathcal{L}_{ij} = \frac{\eta\dot{q}_i}{q_i - q_j + \eta}$$

$$\dot{q}_i = -e^{-\eta p_i}\prod_{k\neq i}^{N}\frac{q_i - q_k + \eta}{q_i - q_k}$$

$$\mathcal{L}_{ij} = \frac{\sinh\eta}{\sinh(q_i - q_j + \eta)}\eta^{-1}\dot{q}_i$$

$$\dot{q}_i = \eta e^{\eta p_i}\prod_{k\neq i}^{N}\frac{\sinh(q_i - q_k - \eta)}{\sinh(q_i - q_k)}$$

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$$\mathcal{H} = \eta^{-1}\sum_{i=1}^{N} e^{-\eta p_i}\prod_{k\neq i}^{N}\frac{q_i - q_k + \eta}{q_i - q_k} \qquad \Big| \qquad \mathcal{H} = \sum_{i=1}^{N} e^{\eta p_i}\prod_{k\neq i}^{N}\frac{\sinh(q_i - q_k - \eta)}{\sinh(q_i - q_k)}$$

$$\mathcal{H} = \mathrm{Tr}\,\mathcal{L}, \quad \mathcal{H}_k = \mathrm{Tr}(\mathcal{L}^k)$$

$$\mathcal{L}_{ij} = \frac{\eta\dot{q}_i}{q_i - q_j + \eta} \qquad \Big| \qquad \mathcal{L}_{ij} = \frac{\sinh\eta}{\sinh(q_i - q_j + \eta)}\eta^{-1}\dot{q}_i$$

$$\dot{q}_i = -e^{-\eta p_i}\prod_{k\neq i}^{N}\frac{q_i - q_k + \eta}{q_i - q_k} \qquad \Big| \qquad \dot{q}_i = \eta e^{\eta p_i}\prod_{k\neq i}^{N}\frac{\sinh(q_i - q_k - \eta)}{\sinh(q_i - q_k)}$$

notice that

$$\mathcal{L} = \dot{Q}\cdot C, \quad Q = \mathrm{diag}(q_1, q_2, \ldots q_N)$$

$$C_{ij} = \frac{\eta}{q_i - q_j + \eta} \qquad \Big| \qquad C_{ij} = \eta^{-1}\frac{\sinh\eta}{\sinh(q_i - q_j + \eta)}$$

Cauchy matrices

Again, rational - XXX

$$L_{ij}^{\mathrm{RS}} = \frac{\nu\,\dot{q}_j}{q_i - q_j + \eta\nu} \qquad \hat{T}^{\mathrm{XXX}}(z) = \mathrm{tr}\,V + \sum_{j=1}^{N} \frac{\hat{H}_j^{\mathrm{XXX}}}{z - z_j}$$

Substitute

$$\nu\eta = \hbar,\ \ q_j = z_j$$

$$\dot{q}_j = \frac{\eta}{\hbar}\,H_j^{\mathrm{XXX}}\Big(\{q_i\}_N;\{\mu_i^1\}_{N_1},\ldots,\{\mu_i^{n-1}\}_{N_{n-1}}\Big)$$

Now the spectrum reads

$$\Big(\underbrace{V_1,\ldots,V_1}_{N-N_1},\underbrace{V_2,\ldots,V_2}_{N_1-N_2},\ldots,\underbrace{V_{n-1},\ldots,V_{n-1}}_{N_{n-2}-N_{n-1}},\underbrace{V_n,\ldots,V_n}_{N_{n-1}}\Big)$$

Trigonometric - XXZ

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admits a factorisation $\quad L^{\mathrm{RS}} = D\,\tilde{V}^{-1}(\epsilon)\tilde{V}(\epsilon - \eta\nu)D^{-1}e^{\eta P}$

$$P = \mathrm{diag}(p_1, ..., p_N), \quad \tilde{V}_{ij}(\epsilon) = \exp\left((2i - 1 - N)(q_j + \epsilon)\right)$$

and $\quad D_{ij} = \delta_{ij}\prod_{k\neq j}^{N}\sinh(q_j - q_k)$

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and $$D_{ij} = \delta_{ij}\prod_{k\neq j}^{N}\sinh(q_j - q_k)$$

The spectral parameter is fictious. Notice also

$\tilde{V}(\epsilon - \eta\nu) = S(\eta\nu)\tilde{V}(\epsilon)$ with

$$\boxed{S_{ij}(\zeta) = \delta_{ij}\exp\left(-(2i - 1 - N)\zeta\right)}$$

Trigonometric - XXZ

$$\hat{T}^{\mathrm{XXZ}}(z) = \mathrm{tr}_0 \left[V_0 \, R_{01}(z - q_1) \ldots R_{0N}(z - q_N) \right]$$

with $\quad V = \mathrm{diag}(V_1, V_2, ..., V_n)$

Trigonometric - XXZ

$$\hat{T}^{\mathrm{xxz}}(z) = \mathrm{tr}_0 \left[V_0\, R_{01}(z - q_1) \ldots R_{0N}(z - q_N) \right]$$

with $V = \mathrm{diag}(V_1, V_2, ..., V_n)$

nested BA gives

$$T^{\mathrm{xxz}}(z) = V_1 \prod_{k=1}^{N} \frac{\sinh(z - q_k + \hbar)}{\sinh(z - q_k)} \prod_{\gamma=1}^{N_1} \frac{\sinh(z - \mu_\gamma^1 - \hbar)}{\sinh(z - \mu_\gamma^1)} +$$

$$+ \sum_{b=2}^{n} V_b \prod_{\gamma=1}^{N_{b-1}} \frac{\sinh(z - \mu_\gamma^{b-1} + \hbar)}{\sinh(z - \mu_\gamma^{b-1})} \prod_{\gamma=1}^{N_b} \frac{\sinh(z - \mu_\gamma^b - \hbar)}{\sinh(z - \mu_\gamma^b)}$$

Trigonometric - XXZ

the BE read

$$V_1 \prod_{k=1}^{N} \frac{\sinh(\mu_\beta^1 - q_k + \hbar)}{\sinh(\mu_\beta^1 - q_k)} = V_2 \prod_{\gamma\neq\beta}^{N_1} \frac{\sinh(\mu_\beta^1 - \mu_\gamma^1 + \hbar)}{\sinh(\mu_\beta^1 - \mu_\gamma^1 - \hbar)}$$

$$V_b \prod_{\gamma=1}^{N_{b-1}} \frac{\sinh(\mu_\beta^b - \mu_\gamma^{b-1} + \hbar)}{\sinh(\mu_\beta^b - \mu_\gamma^{b-1})} = V_{b+1} \prod_{\gamma\neq\beta}^{N_b} \frac{\sinh(\mu_\beta^b - \mu_\gamma^b + \hbar)}{\sinh(\mu_\beta^b - \mu_\gamma^b - \hbar)} \prod_{\gamma=1}^{N_{b+1}} \frac{\sinh(\mu_\beta^b - \mu_\gamma^{b+1} - \hbar)}{\sinh(\mu_\beta^b - \mu_\gamma^{b+1})}$$

Trigonometric - XXZ

transfer matrix pole decomp.

$$\hat{T}^{\mathrm{XXZ}}(z) = \hat{C} + \sum_{k=1}^{N} \hat{H}_i^{\mathrm{XXZ}} \coth(z - q_k) \quad \text{with} \quad \hat{H}_i^{\mathrm{XXZ}} = \mathrm{res}_{z=q_i} \hat{T}^{\mathrm{XXZ}}(z)$$

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their eigenvalues

$$H_i^{\mathrm{XXZ}} = V_1 \, \sinh\hbar \prod_{k\neq i}^{N} \frac{\sinh(q_i - q_k + \hbar)}{\sinh(q_i - q_k)} \prod_{\gamma=1}^{N_1} \frac{\sinh(q_i - \mu_\gamma^1 - \hbar)}{\sinh(q_i - \mu_\gamma^1)}$$

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thus from

$$T^{\mathrm{XXZ}}(\pm\infty) = C \pm \sum_{k=1}^{N} H_k^{\mathrm{XXZ}} = \sum_{a=1}^{n} V_a e^{\pm\hbar M_a}$$

get the ‘sum rules’

$$C = \sum_{a=1}^{n} V_a \cosh(\hbar M_a), \qquad \sum_{k=1}^{N} H_k^{\mathrm{XXZ}} = \sum_{a=1}^{n} V_a \sinh(\hbar M_a)$$

Determinant identity

For the two matrices

$$\mathcal{L}_{ij}(\{x_i\}_N, \{y_i\}_M, g) = \frac{g \sinh \hbar}{\sinh(x_i - x_j + \hbar)} \prod_{k \neq j}^{N} \frac{\sinh(x_j - x_k + \hbar)}{\sinh(x_j - x_k)} \prod_{\gamma=1}^{M} \frac{\sinh(x_j - y_\gamma)}{\sinh(x_j - y_\gamma + \hbar)}$$

with $i, j = 1, \dots, N$

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M, \{x_i\}_N, g) = \frac{g \sinh \hbar}{\sinh(y_\alpha - y_\beta + \hbar)} \prod_{\gamma \neq \beta}^{M} \frac{\sinh(y_\beta - y_\gamma - \hbar)}{\sinh(y_\beta - y_\gamma)} \prod_{k=1}^{N} \frac{\sinh(y_\beta - x_k)}{\sinh(y_\beta - x_k - \hbar)}$$

with $\alpha, \beta = 1, \dots, M$

Determinant identity

For the two matrices

$$\mathcal{L}_{ij}(\{x_i\}_N,\{y_i\}_M,g)=\frac{g\sinh\hbar}{\sinh(x_i-x_j+\hbar)}\prod_{k\neq j}^{N}\frac{\sinh(x_j-x_k+\hbar)}{\sinh(x_j-x_k)}\prod_{\gamma=1}^{M}\frac{\sinh(x_j-y_\gamma)}{\sinh(x_j-y_\gamma+\hbar)}$$

with $i\,,j=1\,,\dots,N$

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M,\{x_i\}_N,g)=\frac{g\sinh\hbar}{\sinh(y_\alpha-y_\beta+\hbar)}\prod_{\gamma\neq\beta}^{M}\frac{\sinh(y_\beta-y_\gamma-\hbar)}{\sinh(y_\beta-y_\gamma)}\prod_{k=1}^{N}\frac{\sinh(y_\beta-x_k)}{\sinh(y_\beta-x_k-\hbar)}$$

with $\alpha,\beta=1\,,\dots,M$

holds true that

$$\det_{N\times N}\Big(\mathcal{L}\left(\{x_i\}_N,\{y_i\}_M,g\right)-\lambda I\Big)=$$

$$=\det_{(N-M)\times(N-M)}(gS-\lambda I)\;\det_{M\times M}\Big(\widetilde{\mathcal{L}}\left(\{y_i\}_M,\{x_i\}_N,g\right)-\lambda I\Big)$$

where

$$S_{ij}(\hbar)=\delta_{ij}\exp\left(-(2i-1-N)\hbar\right)$$

QC-correspondence

Theorem: under identification $\eta\nu = \hbar, \quad \frac{\dot{q}_j}{\eta} = \frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}$

$$\mathrm{Spec}\, L^{\mathrm{RS}}\left(\left\{\dot{q}_j = \eta\,\frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}\right\}_N \{q_j\}_N\,,\,\hbar\right)\Bigg|_{BE} =$$

$$= \Big\{\underbrace{e^{-(N-N_1-1)\hbar}\,V_1,\ldots,e^{(N-N_1-1)\hbar}\,V_1}_{N-N_1},\ \underbrace{e^{-(N_1-N_2-1)\hbar}\,V_2,\ldots,e^{(N_1-N_2-1)\hbar}\,V_2}_{N_1-N_2},\ldots,$$

$$\ldots,\underbrace{e^{-(N_{n-1}-1)\hbar}\,V_n,\ldots,e^{(N_{n-1}-1)\hbar}\,V_n}_{N_{n-1}}\Big\}$$

QC-correspondence

<u>Theorem</u>: under identification $\eta\nu = \hbar, \quad \frac{\dot{q}_j}{\eta} = \frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}$

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$$\ldots,\underbrace{e^{-(N_{n-1}-1)\hbar}\,V_n,\ldots,e^{(N_{n-1}-1)\hbar}\,V_n}_{N_{n-1}}\Big\}$$

can be reformulated as

$$\det\left[L\left(\frac{\eta}{\sinh\hbar}\left\{H_j^{\mathrm{XXZ}}\right\}_N,\{q_j\}_N\,,\,\hbar\right)\Big|_{BE} - \lambda I\right] = \prod_{a=1}^{n}\det\left[V_a\,S_{N_a-N_{a-1}} - \lambda I\right]$$

QC-correspondence

Theorem: under identification $\eta\nu = \hbar, \quad \frac{\dot{q}_j}{\eta} = \frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}$

$$\mathrm{Spec}\, L^{\mathrm{RS}} \left(\left\{ \dot{q}_j = \eta\, \frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar} \right\}_N \{q_j\}_N \,,\, \hbar \right) \Bigg|_{BE} =$$

$$= \Big\{ \underbrace{e^{-(N-N_1-1)\hbar}\, V_1, \ldots, e^{(N-N_1-1)\hbar}\, V_1}_{N-N_1}, \underbrace{e^{-(N_1-N_2-1)\hbar}\, V_2, \ldots, e^{(N_1-N_2-1)\hbar}\, V_2}_{N_1-N_2}, \ldots,$$

$$\ldots, \underbrace{e^{-(N_{n-1}-1)\hbar}\, V_n, \ldots, e^{(N_{n-1}-1)\hbar}\, V_n}_{N_{n-1}} \Big\}$$

can be reformulated as

$$\det\left[L \left(\frac{\eta}{\sinh\hbar} \left\{ H_j^{\mathrm{XXZ}} \right\}_N, \{q_j\}_N \,,\, \hbar \right) \Big|_{BE} - \lambda I \right] = \prod_{a=1}^{n} \det\left[V_a\, S_{N_a - N_{a-1}} - \lambda I \right]$$

Proof: either succesive use of DI, or…

QC-correspondence

or by explicit view of characteristic polynomial coeffs.

$$L = L^{\mathrm{RS}}\left(\left\{\dot{q}_j = \eta\,\frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}\right\}_N \{q_j\}_N\,,\,\hbar\right)$$

QC-correspondence

or by explicit view of characteristic polynomial coeffs.

$$L = L^{\mathrm{RS}}\left(\left\{\dot{q}_j = \eta\,\frac{H_j^{\mathrm{XXZ}}}{\sinh\hbar}\right\}_N \{q_j\}_N\,,\,\hbar\right)$$

coeff of $\det\limits_{N\times N}(\lambda I + A)$ at λ^{N-k} is the sum of $k\times k$ minors

$$\det_{1\le i,j\le k}\frac{\sinh\hbar}{\sinh(q_i - q_j + \hbar)} = \prod_{1\le i,j\le k} C(q_i - q_j), \qquad C(q) = \frac{\sinh^2 q}{\sinh(q+\hbar)\sinh(q-\hbar)}$$

QC-correspondence

or by explicit view of characteristic polynomial coeffs.

$$L = L^{\mathrm{RS}} \left(\left\{ \dot{q}_j = \eta \, \frac{H_j^{\mathrm{XXZ}}}{\sinh \hbar} \right\}_N \{q_j\}_N \, , \, \hbar \right)$$

coeff of $\det\limits_{N\times N}(\lambda I + A)$ at λ^{N-k} is the sum of $k \times k$ minors

$$\det_{1\leq i,j\leq k} \frac{\sinh \hbar}{\sinh(q_i - q_j + \hbar)} = \prod_{1\leq i,j\leq k} C(q_i - q_j), \qquad C(q) = \frac{\sinh^2 q}{\sinh(q+\hbar)\sinh(q-\hbar)}$$

so for $\det\limits_{N\times N}(\lambda I + L) = \sum\limits_{k=0}^{N} J_k \lambda^{N-k}$ get

$$J_k = (\sinh \hbar)^{-k} \sum_{1\leq i_1<\ldots<i_k\leq N} H_{i_1}^{\mathrm{XXZ}} \ldots H_{i_n}^{\mathrm{XXZ}} \prod_{1\leq \alpha<\beta\leq k} C(q_{i_\alpha} - q_{i_\beta})$$

QC-correspondence

or by explicit view of characteristic polynomial coeffs.

$$L = L^{\mathrm{RS}}\left(\left\{\dot{q}_j = \eta\,\frac{H_j^{\mathrm{xxz}}}{\sinh\hbar}\right\}_N \{q_j\}_N\,,\,\hbar\right)$$

coeff of $\det\limits_{N\times N}(\lambda I + A)$ at λ^{N-k} is the sum of $k\times k$ minors

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so for $\det\limits_{N\times N}(\lambda I + L) = \sum\limits_{k=0}^{N} J_k \lambda^{N-k}$ get

$$J_k = (\sinh\hbar)^{-k}\sum_{1\leq i_1<\ldots<i_k\leq N} H_{i_1}^{\mathrm{xxz}}\ldots H_{i_n}^{\mathrm{xxz}}\prod_{1\leq\alpha<\beta\leq k} C(q_{i_\alpha} - q_{i_\beta})$$

so finally

$$\boxed{\sum_{1\leq i_1<\ldots<i_k\leq N} H_{i_1}^{\mathrm{xxz}}\ldots H_{i_k}^{\mathrm{xxz}}\prod_{1\leq\alpha<\beta\leq k} C(q_{i_\alpha} - q_{i_\beta}) = (\sinh\hbar)^k\sum_{1\leq i_1<\ldots<i_k\leq N}\lambda_{i_1}\ldots\lambda_{i_k}}$$

QC-correspondence

$$k = N$$

$$\prod_{j=1}^{N} H_j^{\mathrm{xxz}} \cdot \prod_{1 \leq l < m \leq N} C(q_l - q_m) = (\sinh \hbar)^N \prod_{a=1}^{n} V_a^{M_a}$$

the BEs give

$$\prod_{k=1}^{N} H_k^{\mathrm{xxz}} = (V_1 \sinh \hbar)^N \prod_{i=1}^{N} \prod_{k \neq i}^{N} \frac{\sinh(q_i - q_k + \hbar)}{\sinh(q_i - q_k)} \prod_{i=1}^{N} \prod_{\gamma=1}^{N_1} \frac{\sinh(q_i - \mu_\gamma^1 - \hbar)}{\sinh(q_i - \mu_\gamma^1)}$$

first double prod. cancels the prod. of C-factors

QC-correspondence

$$k = N$$

$$BE_1: \quad V_1^{N_1} \prod_{\beta=1}^{N_1} \prod_{k=1}^{N} \frac{\sinh(q_k - \mu_\beta^1 - \hbar)}{\sinh(q_k - \mu_\beta^1)} = V_2^{N_1} \prod_{\beta=1}^{N_1} \prod_{\gamma=1}^{N_2} \frac{\sinh(\mu_\beta^1 - \mu_\gamma^2 - \hbar)}{\sinh(\mu_\beta^1 - \mu_\beta^2)}$$

$$BE_b: \; V_b^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b-1}} \frac{\sinh(\mu_\gamma^{b-1} - \mu_\beta^b - \hbar)}{\sinh(\mu_\gamma^{b-1} - \mu_\beta^b)} = V_{b+1}^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b+1}} \frac{\sinh(\mu_\beta^b - \mu_\gamma^{b+1} - \hbar)}{\sinh(\mu_\beta^b - \mu_\gamma^{b+1})}$$

$$BE_{n-1}: \quad V_{n-1}^{N_{n-1}} \prod_{\beta=1}^{N_{n-1}} \prod_{\gamma=1}^{N_{n-2}} \frac{\sinh(\mu_\gamma^{n-2} - \mu_\beta^{n-1} - \hbar)}{\sinh(\mu_\gamma^{n-2} - \mu_\beta^{n-1})} = V_n^{N_{n-1}}$$

QC-correspondence

$$k = N$$

$$BE_1 : \quad V_1^{N_1} \prod_{\beta=1}^{N_1} \prod_{k=1}^{N} \frac{\sinh(q_k - \mu^1_\beta - \hbar)}{\sinh(q_k - \mu^1_\beta)} = V_2^{N_1} \prod_{\beta=1}^{N_1} \prod_{\gamma=1}^{N_2} \frac{\sinh(\mu^1_\beta - \mu^2_\gamma - \hbar)}{\sinh(\mu^1_\beta - \mu^2_\beta)}$$

$$BE_b : \; V_b^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b-1}} \frac{\sinh(\mu^{b-1}_\gamma - \mu^b_\beta - \hbar)}{\sinh(\mu^{b-1}_\gamma - \mu^b_\beta)} = V_{b+1}^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b+1}} \frac{\sinh(\mu^b_\beta - \mu^{b+1}_\gamma - \hbar)}{\sinh(\mu^b_\beta - \mu^{b+1}_\gamma)}$$

$$BE_{n-1} : \quad V_{n-1}^{N_{n-1}} \prod_{\beta=1}^{N_{n-1}} \prod_{\gamma=1}^{N_{n-2}} \frac{\sinh(\mu^{n-2}_\gamma - \mu^{n-1}_\beta - \hbar)}{\sinh(\mu^{n-2}_\gamma - \mu^{n-1}_\beta)} = V_n^{N_{n-1}}$$

this chain of identities thus proves

$$\prod_{j=1}^{N} H_j^{\mathrm{XXZ}} \cdot \prod_{1 \le l < m \le N} C(q_l - q_m) = (\sinh \hbar)^N \prod_{a=1}^{n} V_a^{M_a}$$

QC-correspondence

$$k = 1$$

we have

$$\sum_{j=1}^{N} H_j^{\mathrm{xxz}} = \sinh\hbar \sum_{j=1}^{N} \lambda_j, \qquad \lambda_j \in \operatorname{Spec} L$$

since

$$\sum_{j=0}^{N_{b-1}-N_b-1} e^{-(N_{b-1}-N_b-1)+2j\hbar} = \frac{\sinh\big(\hbar(N_{b-1}-N_b)\big)}{\sinh\hbar}$$

which is exactly the ‘sum rule’

Beyond XXZ - trigonometric RS

- Non-relativistic limit of XXZ - the Gaudin model and the limit of RS - the Calogero-Moser model is also studied. Determinant identity holds, spectral correspondence is present.

- Free XX limit of $\hbar \to i\pi/2$ is mentioned. The Bethe equations separate in this limit, the whole system can be mapped onto non-interacting fermions on a line

- Generalisation to XYZ feels troublesome

- Connection(?) to:

- A. Mironov, A. Morozov, B. Runov, Y. Zenkevich, A. Zotov, *Spectral dualities in XXZ spin chains and five dimensional gauge theories,* JHEP 1312 (2013) 034, arXiv:1307.1502 [hep-th]

- A. Mironov, A. Morozov, B. Runov, Y. Zenkevich, A. Zotov, *Spectral Duality Between Heisenberg Chain and Gaudin Model,* Letters in Mathematical Physics: Volume 103, Issue 3 (2013), Page 299-329, arXiv:1206.6349 [hep-th]

Thank you for attention!