Pontryagin algebras and LS-category of moment-angle complexes in the flag case

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Plan

- Preliminaries
- $\mathbb{Z} \times \mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk)$ and $H_*(\Omega \mathcal{Z}_\mathcal{K}; \Bbbk)$
- Number of relations in the minimal presentation of $H_*(\Omega \mathcal{Z}_\mathcal{K}; \Bbbk)$
- Milnor-Moore spectral sequence for $\mathcal{Z}_\mathcal{K}$ in the flag case
- $\mathrm{cat}(\mathcal{Z}_\mathcal{K}) \geq \dots$ in terms of $\mathcal{K}^f$
- $\mathrm{cat}(\mathcal{Z}_\mathcal{K}) = \dots$ for flag $\mathcal{K}$

Preliminaries: simplicial complexes

A simplicial complex $\mathcal{K}$ on vertex set V is a collection of subsets $I \subset V$ that is closed under inclusions ($I \subset J \in \mathcal{K} \Rightarrow I \in \mathcal{K}$). We assume that $\mathcal{K}$ is without ghost vertices, so $\{i\} \in \mathcal{K}$ for all $i \in V$. Note that $\varnothing \in \mathcal{K}$.

Missing faces

$I \notin \mathcal{K}$ is a missing face for $\mathcal{K}$ if every $J \subsetneq I$ belongs to $\mathcal{K}$. The set of missing faces is denoted by $\mathrm{MF}(\mathcal{K})$.

Flagness and flagification

$\mathcal{K}$ is flag if all its missing faces are edges. For every $\mathcal{K}$, there is unique flag complex $\mathcal{K}^f$ such that $\mathrm{sk}_1\,\mathcal{K} = \mathrm{sk}_1\,\mathcal{K}^f$. Clearly, $\mathcal{K} \subset \mathcal{K}^f$.

Links and full subcomplexes

Given $J \subset V$, define $\mathcal{K}_J := \{I \in \mathcal{K} :\ I \subset J\}$.
Given $I \in \mathcal{K}$, define $\mathrm{lk}_{\mathcal{K}}\, I := \{J \subset V :\ I \sqcup J \in \mathcal{K}\}$. Note that $\mathrm{lk}_{\mathcal{K}}\,\varnothing = \mathcal{K}$.
If $\mathcal{K}$ is flag then *links are full subcomplexes*.

Preliminaries: polyhedral products

Let $\mathcal{K}$ be a simplicial complex on vertex set $[m] := \{1, \dots, m\}$.

Polyhedral product functor

For (X, A) a pair of spaces,

$$(X, A)^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \prod_{i \in I} X \times \prod_{i \notin I} A \subset X^m.$$

Main examples

- Davis-Januszkiewicz space $\mathrm{DJ}(\mathcal{K}) := (\mathbb{C}\mathrm{P}^\infty, \mathrm{pt})^{\mathcal{K}}$;
- Moment-angle complex $\mathcal{Z}_{\mathcal{K}} := (D^2, S^1)^{\mathcal{K}}$;
- Real moment-angle complex $\mathcal{R}_{\mathcal{K}} := (D^1, S^0)^{\mathcal{K}}$.

Theorem (Buchstaber-Panov, 2000)

There is a homotopy fibration $\mathcal{Z}_{\mathcal{K}} \to \mathrm{DJ}(\mathcal{K}) \to (\mathbb{C}\mathrm{P}^\infty)^m$.

Preliminaries: face rings

Unless specified, $\Bbbk$ is an associative ring with unit, $\mathcal{K}$ is a simplicial complex on vertex set $[m]$.

Face ring

$$\Bbbk[\mathcal{K}] := \Bbbk[v_1, \dots, v_m]/(v_{i_1} \cdot \ldots \cdot v_{i_k} : \{i_1, \dots, i_k\} \notin \mathcal{K}).$$

There is a $\mathbb{Z}^m_{\geq 0}$-grading: $\deg v_i := 2e_i$. Clearly, $e_i \mapsto 1$ makes it a $\mathbb{Z}$-grading. We identify $J \subset [m]$ with $\sum_{j\in J} e_j \in \mathbb{Z}^m_{\geq 0}$.

Face ring in the flag case

If $\mathcal{K}$ is flag then $\Bbbk[\mathcal{K}]$ is quadratic:

$$\Bbbk[\mathcal{K}] = T(v_1, \dots, v_m)/(v_i v_j - v_j v_i,\ i < j;\ v_i v_j = 0,\ \{i,j\} \notin \mathcal{K}).$$

Fröberg (1975) has shown that $\Bbbk[\mathcal{K}]$ is Koszul if $\mathcal{K}$ is flag.
(A connected associative $\Bbbk$-algebra A is Koszul iff its *quadratic dual* $A^!$ coincides with the Yoneda algebra $\operatorname{Ext}_A(\Bbbk, \Bbbk)$.)

Preliminaries: cohomology rings of polyhedral products

Theorem (Baskakov-Buchstaber-Panov — Franz, 2004)

- $H^*(\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \Bbbk[\mathcal{K}]$ and hence is $\mathbb{Z}^m_{\geq 0}$-graded:

$$H^n(\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \bigoplus_{\alpha:\ |2\alpha|=n} H^{2\alpha}(\mathrm{DJ}(\mathcal{K});\Bbbk),\ \ H^{2\alpha}(\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \Bbbk[\mathcal{K}]_{2\alpha}.$$

- $H^*(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \mathrm{Tor}^{\Bbbk[v_1,\dots,v_m]}(\Bbbk[\mathcal{K}],\Bbbk)$ and hence is $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded:

$$H^n(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \bigoplus_{n=2|\alpha|-i} H^{-i,2\alpha}(\mathcal{Z}_\mathcal{K};\Bbbk),$$

$$H^{-i,2\alpha}(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \mathrm{Tor}_i^{\Bbbk[v_1,\dots,v_m]}(\Bbbk[\mathcal{K}],\Bbbk)_{2\alpha}.$$

- ("Hochster's formula") $H^{-i,2\alpha}(\mathcal{Z}_\mathcal{K};\Bbbk) = 0$ if $\alpha \neq J$, and $H^{-i,2J}(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J;\Bbbk)$.

+Similar formulas for $H_*(\mathrm{DJ}(\mathcal{K});\Bbbk)$ and $H_*(\mathcal{Z}_\mathcal{K};\Bbbk)$ in terms of $\Bbbk\langle\mathcal{K}\rangle$.

Pontryagin algebra of the Davis-Januszkiewicz space

Consider the $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded $\Bbbk$-algebra, $\deg u_i := (-1, 2e_i)$:

$\Bbbk[\mathcal{K}]^! := T(u_1,\dots,u_m)/(u_i^2=0,\ i=1\dots m;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K}).$

This ring is quadratic dual to $\Bbbk[\mathcal{K}]$.

Theorem (Panov-Ray, 2007+ V., 2021)

- $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)\cong\mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)$ and hence is $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded:

$$H_n(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)\cong\bigoplus_{n=2|\alpha|-i} H_{-i,2\alpha}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk),$$

$$H_{-i,2\alpha}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)\cong\mathrm{Ext}^i_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)_{2\alpha}.$$

- $\Bbbk[\mathcal{K}]^!\hookrightarrow H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$.
- $\Bbbk[\mathcal{K}]^!\cong H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ if $\mathcal{K}$ is flag. (This is just Koszulness of $\Bbbk[\mathcal{K}]$.)

(Original statement was without multigrading, only for field coefficients. But the proof remains valid for arbitrary ring $\Bbbk$.)

Preliminaries: the split fibration of H-spaces

(Panov-Ray, 2007) Loop the fibration $\mathcal{Z}_\mathcal{K} \to \mathrm{DJ}(\mathcal{K}) \to (\mathbb{C}\mathrm{P}^\infty)^m$ to obtain a principal fibration of H-spaces

$$\Omega\mathcal{Z}_\mathcal{K} \xrightarrow{i} \Omega\mathrm{DJ}(\mathcal{K}) \xrightarrow{p} \Omega(\mathbb{C}\mathrm{P}^\infty)^m \cong \mathbb{T}^m.$$

It has a section:

$$\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_\mathcal{K} \times \mathbb{T}^m \text{ (not as H-spaces!).}$$

In particular, $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk) \hookrightarrow H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ and

$$H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk) \otimes \Lambda[u_1,\dots,u_m]$$

as $\Bbbk$-modules and as graded left $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$-modules (not as algebras!). We saw that $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ is actually $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded. This induces the $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$.

$H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$: known results in the flag case

Let $\Bbbk$ be a field.

- (Panov-Ray, 2007) Let $h(t)$ be the *h-polynomial* for $\mathcal{K}$. Then

$$\sum_{k\geq 0}\dim H_k(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)t^k = \frac{1}{(1+t)^{m-\dim\mathcal{K}}h(-t)}.$$

- (Grbic-Panov-Theriault-Wu, 2012) An explicit set of $\sum_{J\subset[m]}\dim \widetilde{H}_0(\mathcal{K}_J;\Bbbk)$ minimal generators for $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$; $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is a free algebra iff $\mathrm{sk}_1\,\mathcal{K}$ is a *chordal* graph.
- (Veryovkin, 2015) Full description of the one-relator algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, where $\mathcal{K}$ is a p-cycle, $p=4,5,6$.
- (Grbic-Ilyasova-Panov-Simmons, 2020) $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is a one-relator algebra iff $\mathcal{K}=C_p*\Delta^q$, where $p\geq 4$ and C_p is a p-cycle.
- (Cai, 2021 – work in progress) Every loop in $\mathcal{K}_J$ gives a relation in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ of multidegree $(-|J|,2J)$.

An explicit description of defining relations in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ seems difficult, but we can calculate their number.

$\mathrm{Tor}^A(\Bbbk, \Bbbk) \leftrightarrow$ generators and relations of A

Let A be a connected graded algebra over a field $\Bbbk$. Presentations $A \cong T(a_1, \ldots, a_N)/(r_1, \ldots, r_M)$ correspond bijectively to exact sequences

$$\bigoplus_{j=1}^{M} A \cdot r_j \to \bigoplus_{i=1}^{N} A \cdot a_i \to A \xrightarrow{\varepsilon} \Bbbk \to 0$$

of left A-modules.

Taking the minimal resolution of the left A-module $\Bbbk$, we obtain a minimal presentation $A \cong T(a_1, \ldots, a_N)/(r_1, \ldots, r_M)$, such that

$$\left(\mathrm{Ext}^1_A(\Bbbk, \Bbbk)\right)^{\#} \cong \mathrm{Tor}^A_1(\Bbbk, \Bbbk) \cong \bigoplus_{i=1}^{N} \Bbbk \cdot a_i,$$

$$\left(\mathrm{Ext}^2_A(\Bbbk, \Bbbk)\right)^{\#} \cong \mathrm{Tor}^A_2(\Bbbk, \Bbbk) \cong \bigoplus_{j=1}^{M} \Bbbk \cdot r_j$$

as graded $\Bbbk$-modules.

(Reference: §7 in C.T.C.Wall, *Generators and Relations for the Steenrod Algebra*, Ann. of Math. 72(3), 429-444).)

$\mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk) \leftarrow$ generators and relations of $\Bbbk[\mathcal{K}]$

Example

$\Bbbk[\mathcal{K}]$ has a minimal presentation with generators $v_1,\dots,v_m$ and relations

$$v_iv_j - v_jv_i = 0,\ \forall i>j;\ \ v_{j_1}v_{j_2}\dots v_{j_k}=0,\ \{j_1<\dots<j_k\}\in\mathrm{MF}(\mathcal{K}).$$

Hence:

- Generators $v_i \in \Bbbk[\mathcal{K}]$ correspond to elements $u_i \in \mathrm{Ext}^1_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)_{2e_i}$ and then to generators $u_i \in H_{-1,2e_i}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$;
- Every $J\in\mathrm{MF}(\mathcal{K})$ gives an element $\mu_J \in \mathrm{Ext}^2_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)_{2J}$ that can be interpreted as a "higher product" $[u_{j_1},\dots,u_{j_k}] \in H_{-2,2J}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$.
- In particular, missing edges $\{i,j\}$ give ordinary products $u_iu_j \in H_{-2,2e_i+2e_j}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$;
- Commutativity relations $v_iv_j - v_jv_i$ give ordinary products u_ju_i.

Thus we described $H_{-i,2*}(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk),\ i\le 2$.

$\mathrm{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K})}(\Bbbk,\Bbbk)\to$ generators and relations of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$

Theorem (V., 2021)

For flag $\mathcal{K}$ without ghost vertices,

- $\mathrm{Tor}_n^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)\cong\bigoplus_{J\subset[m]}\widetilde{H}_{n-1}(\mathcal{K}_J;\Bbbk)$;
- More precisely, $\mathrm{Tor}_n^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)_{-|J|,2J}\cong\widetilde{H}_{n-1}(\mathcal{K}_J;\Bbbk)$;
- If $\Bbbk$ is a field, the minimal presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ has $\sum_{J\subset[m]}\dim\widetilde{H}_1(\mathcal{K}_J;\Bbbk)$ relations between $\sum_{J\subset[m]}\dim\widetilde{H}_0(\mathcal{K}_J;\Bbbk)$ generators: for every $J\subset[m]$, exactly $\dim\widetilde{H}_0(\mathcal{K}_J;\Bbbk)$ generators and $\dim\widetilde{H}_1(\mathcal{K}_J;\Bbbk)$ relations of multidegree $(-|J|,2J)$;
- As formal power series,

$$\sum_{i,\alpha}\dim H_{-i,2\alpha}(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)t^{-i}\lambda^{2\alpha}=-\frac{1}{\sum_{J\subset[m]}\widetilde{\chi}(\mathcal{K}_J)t^{-|J|}\lambda^{2J}}.$$

(We write $\widetilde{\chi}(X):=\chi(X)-1$, $\lambda^\alpha:=\prod_{j=1}^m\lambda_j^{\alpha_j}$.)

$\mathrm{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)$: sketch of the calculation

Let $\mathcal{K}$ be flag. For a class of quadratic $\Bbbk$-algebras A, Fröberg (1975) constructed free resolutions of the left A-module $\Bbbk$. In particular, for $A=\Bbbk[\mathcal{K}]^!$ we have

$$\dots\to\Bbbk[\mathcal{K}]^!\otimes\Bbbk\langle\mathcal{K}\rangle_4\to\Bbbk[\mathcal{K}]^!\otimes\Bbbk\langle\mathcal{K}\rangle_2\to\Bbbk[\mathcal{K}]^!\to\Bbbk\to 0.$$

Since $\Bbbk[\mathcal{K}]^!\cong H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\otimes\Lambda[m]$ as a left $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$-module, it is an exact sequence of left $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$-modules:

$$\dots\to H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\otimes(\Lambda[m]\otimes\Bbbk\langle\mathcal{K}\rangle_2)\to H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\otimes\Lambda[m]\to\Bbbk\to 0.$$

So $\mathrm{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk,\Bbbk)\cong H(\Lambda[m]\otimes\Bbbk\langle\mathcal{K}\rangle,d)$, where d (accidentally?) coincides with the Koszul differential. The rest is well known.

Milnor-Moore (=Rothenberg-Steenrod) spectral sequence

Let X be a 1-connected space, $\mathbb{k}$ be a field. There is a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{H_*(\Omega X;\mathbb{k})}(\mathbb{k},\mathbb{k})_q \Rightarrow H_{p+q}(X;\mathbb{k}).$$

The Toomer invariant: $e_{\mathbb{k}}(X) := \max\{p : E^\infty_{p,*} \neq 0\}$.

Theorem (V., 2021)

Let $\mathcal{K}$ be a flag simplicial complex. Then

- The Milnor-Moore spectral sequence for $\mathcal{Z}_{\mathcal{K}}$ collapses at E^2.
- $e_{\mathbb{k}}(\mathcal{Z}_{\mathcal{K}}) = 1 + \max_{J\subset[m]} \mathrm{hdim}_{\mathbb{k}}\, \mathcal{K}_J$.

(Here $\mathrm{hdim}_{\mathbb{k}}\, X := \max\{i : \widetilde{H}_i(X;\mathbb{k}) \neq 0\}$.)

Proof.

- $\dim E^\infty = \dim H_*(\mathcal{Z}_{\mathcal{K}};\mathbb{k}) = \dim \mathrm{Tor}^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbb{k})}(\mathbb{k},\mathbb{k}) = \dim E^2$.
- $E^\infty_{p,*} = E^2_{p,*} = \bigoplus_{J\subset[m]} \widetilde{H}_{p-1}(\mathcal{K}_J;\mathbb{k})$. □

A lower bound on $\mathrm{cat}(\mathcal{Z}_\mathcal{K})$ in the flag case

Lusternik-Schnirelmann category

Let X be a connected CW-complex. $\mathrm{cat}(X) \leq n$ iff there is an open covering $X = U_0 \cup \cdots \cup U_n$ such that all $U_i \hookrightarrow X$ are null-homotopic.

By a result of Ginsburg (1962), $\mathrm{cat}(X) \geq e_\Bbbk(X)$ for any field $\Bbbk$.

Proposition (V., 2021)

If $\mathcal{K}$ is flag then $\mathrm{cat}(\mathcal{Z}_\mathcal{K}) \geq 1 + \max_{J \subset [m]} \mathrm{cdim}_\mathbb{Z} \mathcal{K}_J$.

Proof.

Use the univ. coeff. formula and $e_\Bbbk(\mathcal{Z}_\mathcal{K}) = 1 + \max_{J \subset [m]} \mathrm{hdim}_\Bbbk \mathcal{K}_J$. □

A lower bound on $\mathrm{cat}(\mathcal{Z}_\mathcal{K})$ in terms of $\mathcal{K}^f$

Beben and Grbic (2016) gave various lower and upper bounds on $\mathrm{cat}(\mathcal{Z}_\mathcal{K})$.

Proposition (Beben-Grbic, 2016)

Let $\mathcal{K}_0 \subset \cdots \subset \mathcal{K}_s$ be a filtration of simplicial complexes, such that $I \subsetneq J \in \mathcal{K}_{j+1}$ implies $I \in \mathcal{K}_j$. Then $\mathrm{cat}(\mathcal{Z}_{\mathcal{K}_s}) \leq \mathrm{cat}(\mathcal{Z}_{\mathcal{K}_0}) + s$.

Let $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_{j+1} = \mathcal{K}_j \cup \mathrm{MF}(\mathcal{K}_j) \setminus \{\text{missing edges of } \mathcal{K}_j\}$. This filtration satisfies the condition above, and stabilises at certain $\mathcal{K}_\nu = \mathcal{K}^f$.

Corollary (V., 2021)

Denote $d := \max_{J \subset [m]} \mathrm{cdim}_\mathbb{Z} \mathcal{K}^f_J$. Then $\mathrm{cat}(\mathcal{Z}_\mathcal{K}) \geq d - \nu + 1$.

The number $\nu \geq 0$ measures the *"non-flagness"* of $\mathcal{K}$: $\nu = 0$ iff $\mathcal{K}$ is flag; if $\mathrm{sk}_i \mathcal{K} = \mathrm{sk}_i \mathcal{K}^f$ then $\nu \leq \dim \mathcal{K}^f - i$.
So this bound is the most useful for "nearly flag" $\mathcal{K}$.

cat($\mathcal{R}_\mathcal{K}$) in the flag case

Proposition (Beben-Grbic, 2016)

$\mathrm{cat}((\Sigma X, \Sigma A)^\mathcal{K}) \leq \mathrm{cat}((X, A)^\mathcal{K})$. In particular, $\mathrm{cat}(\mathcal{Z}_\mathcal{K}) \leq \mathrm{cat}(\mathcal{R}_\mathcal{K})$.

Theorem (Davis)

If $\mathcal{K}$ is flag then $\mathcal{R}_\mathcal{K} = B(\mathrm{RC}'_\mathcal{K})$, where $\mathrm{RC}_\mathcal{K}$ is the *right-angled Coxeter group* associated with the graph $\mathrm{sk}_1\,\mathcal{K}$.

Theorem (Eilenberg-Ganea, 1957)

$\mathrm{cat}(BG) = \mathrm{cd}\, G$. (Stallings-Swan theorem covers the case $\mathrm{cd}\, G = 1$.)

(Here $\mathrm{cd}\, G := \max\{i : H^i(G; M) \neq 0$ for some G-module $M\}$.)
It follows: if $\mathcal{K}$ is flag then

$$\mathrm{cat}(\mathcal{R}_\mathcal{K}) = \mathrm{cd}\,\mathrm{RC}'_\mathcal{K} \ (= \mathrm{vcd}\,\mathrm{RC}_\mathcal{K}.)$$

Dranishnikov's formula

vcd of Coxeter groups was studied by Bestvina (1993), Davis (1995)...

Theorem (Dranishnikov, 1997)

$(\mathrm{cd}\,\mathrm{RC}'_{\mathcal{K}} =)$ $\mathrm{vcd}\,\mathrm{RC}_{\mathcal{K}} = 1 + \max_{I\in\mathcal{K}} \mathrm{cdim}_{\mathbb{Z}}\,\mathrm{lk}_{\mathcal{K}}\, I$ for flag $\mathcal{K} \neq \Delta^{m-1}$.

Thus we obtain:

$$1 + \max_{J\subset[m]} \mathrm{cdim}_{\mathbb{Z}}\,\mathcal{K}_J \leq \mathrm{cat}(\mathcal{Z}_{\mathcal{K}}) \leq \mathrm{cat}(\mathcal{R}_{\mathcal{K}}) = \mathrm{vcd}\,\mathrm{RC}_{\mathcal{K}} = 1 + \max_{I\in\mathcal{K}} \mathrm{cdim}_{\mathbb{Z}}\,\mathrm{lk}_{\mathcal{K}}\, I.$$

But $\mathrm{lk}_{\mathcal{K}}\, I$ is a full subcomplex of $\mathcal{K}$ in the flag case! Hence

$$\max_{I\in\mathcal{K}} \mathrm{cdim}_{\mathbb{Z}}\,\mathrm{lk}_{\mathcal{K}}\, I \leq \max_{J\subset[m]} \mathrm{cdim}_{\mathbb{Z}}\,\mathcal{K}_J,$$

and all "$\leq$" above are "$=$".

Final result on $\mathrm{cat}(\mathcal{Z}_\mathcal{K})$ and $\mathrm{cat}(\mathcal{R}_\mathcal{K})$ in the flag case

Theorem (V., 2021)

Let $\mathcal{K} \neq \Delta^{m-1}$ be a flag simplicial complex on $[m]$ without ghost vertices. Then

$$\mathrm{vcd}\,\mathrm{RC}_\mathcal{K} = \mathrm{cd}\,\mathrm{RC}'_\mathcal{K} = \mathrm{cdim}_\mathbb{Z}\,\mathcal{R}_\mathcal{K} = \mathrm{cat}(\mathcal{R}_\mathcal{K}) = \mathrm{cat}(\mathcal{Z}_\mathcal{K}) =$$
$$= 1 + \max_{J\subset[m]} \mathrm{cdim}_\mathbb{Z}\,\mathcal{K}_J = 1 + \max_{I\in\mathcal{K}} \mathrm{cdim}_\mathbb{Z}\,\mathrm{lk}_\mathcal{K}\, I. \quad \square$$

Remark

For flag complexes, we obtained the identity that, in fact, holds for all $\mathcal{K}$:

$$\max_{J\subset[m]} \mathrm{cdim}_\mathbb{Z}\,\mathcal{K}_J = \max_{I\in\mathcal{K}} \mathrm{cdim}_\mathbb{Z}\,\mathrm{lk}_\mathcal{K}\, I.$$

This is Alexander dual to the results of Ayzenberg (2012). He proved that s-link-acyclic simplicial complexes coincide with s-subcomplex-acyclic. (Alexander duality: $\widetilde{H}_i(\mathcal{K}_J) \cong \widetilde{H}^{|J|-3-i}(\mathrm{lk}_{\widehat{\mathcal{K}}}\,\widehat{J})$.)

Further questions

The questions are mainly about the Milnor-Moore spectral sequence for $\mathcal{Z}_{\mathcal{K}}$. It seemingly admits the additional $\mathbb{Z}^m_{\geq 0}$-grading.

- Coformality implies $E^2 \cong E^\infty$. Is $\mathcal{Z}_{\mathcal{K}}$ coformal for flag $\mathcal{K}$?
- Any examples of non-trivial differentials?
  ("Are there Massey products in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$?")
- Is there a good description of E^∞?
  ("What is the filtration on $H_*(\mathcal{Z}_{\mathcal{K}})$?")
- In the flag case, $H_*(\mathcal{Z}_{\mathcal{K}};\mathbb{k}) \cong \mathrm{Tor}^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbb{k})}(\mathbb{k},\mathbb{k})$ as $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded $\mathbb{k}$-modules. Is this isomorphism comultiplicative?
- Understand the A_∞-structure on $\mathrm{Ext}_{\mathbb{k}[\mathcal{K}]}(\mathbb{k},\mathbb{k}) \cong H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbb{k})$...
- What about general $(\mathrm{cone}(A), A)^{\mathcal{K}}$?

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