Dichotomy theorem in computational social choice theory

N. L. Polyakov, Higher School of Economics

We consider some problems of aggregation of individual preferences. We show that under rather general assumptions there are only two clones of aggregation rules that allow invariant symmetric classes of preferences, each of these clones being generated by a single function.

Let A be a finite set and r a natural number. The symbol $[A]^r$ denotes the set of all r-element subsets of A. *Individual preferences* are modeled by *r-choice functions* on a set A, i.e. functions $\mathfrak{c} : [A]^r \to A$ satisfying $f(p) \in p$ for any $p \in [A]^r$. The set of all r-choice functions on a set A is denoted by $\mathfrak{C}_r(A)$. A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is called *symmetric* if $\mathfrak{c} \in \mathfrak{D} \Rightarrow \mathfrak{c}_\sigma \in \mathfrak{D}$ for any permutation σ of A where $\mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p)$ for any $p \in [A]^r$. A (simple local) *aggregation rule* is a function $f : A^n_{\leq r} \to A$ where $A^n_{\leq r} = \{\mathbf{a} \in A^n : |\mathrm{ran}\,\mathbf{a}| \leq r\}$, see [1] (cf [2]). For all $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n \in \mathfrak{C}_r(A)$ and $f : A^n_{\leq r} \to A$ the symbol $f(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n)$ denotes the r-choice function $\mathfrak{c}$ defined by $\mathfrak{c}(p) = f(\mathfrak{c}_1(p), \mathfrak{c}_2(p), \dots, \mathfrak{c}_n(p))$ for all $p \in [A]^r$. An aggregation rule $f : A^n_{\leq r} \to A$ *preserves* a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if $f(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in \mathfrak{D}$ for all $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n \in \mathfrak{D}$. The Galois connection generated in natural sense by the preservation relation is denoted $(\mathrm{Inv}_r, \mathrm{Pol}_r)$. A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ has the *Arrow property* if $\mathrm{Pol}_r(\mathfrak{D})$ contains only projections (dictatorship rules). All symmetric sets without the Arrow property were classified in [3], see also [1]. In addition, it is shown [4] that if $r = 2$ (this is the most important case) and $|A| \geq 5$ then for any set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ without the Arrow property the set $\mathrm{Pol}_r(\mathfrak{D})$ consists of functions generated by the "counting-out game" function ℓ defined by $\ell(x, y, y) = \ell(y, x, y) = \ell(y, y, x) = x$. This result can be considered as a generalization of Arrow's impossibility theorem [5]. In essence, it means that there are no acceptable aggregation rules for symmetric sets of preferences.

For positive results, we consider a more general situation. A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is called *trivial* if $\mathfrak{D} = \{\mathfrak{c} \in \mathfrak{C}_r(A) : \mathfrak{c} \restriction_B = \mathfrak{d} \restriction_B\}$ for some $\mathfrak{d} \in \mathfrak{C}_r(A)$ and $B \subseteq [A]^r$ (a trivial set $\mathfrak{D}$ is preserved by any aggregation rule). A set $\mathbb{D} \subseteq \mathscr{P}(\mathfrak{C}_r(A))$ is called trivial if it contains only trivial sets. A set $\mathbb{D} \subseteq \mathscr{P}(\mathfrak{C}_r(A))$ is called *symmetric* if $\mathfrak{D} \in \mathbb{D} \Rightarrow \mathfrak{D}_\sigma \in \mathbb{D}$ for any permutation σ of A where $\mathfrak{D}_\sigma = \{\mathfrak{c}_\sigma : \mathfrak{c} \in \mathfrak{D}\}$ (for example, the class of all *single-peaked* domain [6] is symmetric). Let $\partial : A^3_{\leq 2} \to A$ be a *majority* function. We prove the following dichotomy theorems.

Theorem 1. *Let $|A| \geq 5$. Let $f : A^n_{\leq 2} \to A$ be a non-dictatorship aggregation rule and $\mathbb{D} \subseteq \mathrm{Inv}_2(f)$ a non-trivial symmetric set. Then $\mathbb{D} \subseteq \mathrm{Inv}_2(\partial)$ or $\mathbb{D} \subseteq \mathrm{Inv}_2(\ell)$*

Let $f : A^n_{\leq r} \to A$ and $\mathfrak{C} \subseteq \mathfrak{C}_r(A)$. A set $\mathfrak{D}$ is *compatible with the pair* $(f, \mathfrak{C})$ if $\mathfrak{D} \subseteq \mathfrak{C}$ and $f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in \mathfrak{C}$ for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}$ A set of all sets what is compatible with $(f, \mathfrak{C})$ is denoted by $\mathrm{Comp}\,(f, \mathfrak{C})$. A function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is called rational if $\mathfrak{c}(p) = \max_{\succ} p$ for some linear order $\succ$ on A. The set of all rational function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is denoted by $\mathfrak{R}_r(A)$.

Theorem 2. *Let* $|A| \geq 5$. *Let* $f : A^n_{\leq 2} \to A$ *be a non-dictatorship aggregation rule and* $\mathbb{C} \subseteq \mathrm{Comp}(f, \mathfrak{R}_2(A))$ *a non-trivial symmetric set. Then there is a symmetric class* $\mathbb{D} \subseteq \mathscr{P}(\mathfrak{R}_2(A))$ *such that*

1. $\mathbb{D} \subseteq \mathrm{Inv}_2(\partial) \cap \mathrm{Inv}_2(f)$ *or* $\mathbb{D} \subseteq \mathrm{Inv}_2(\ell) \cap \mathrm{Inv}_2(f)$ *and*

2. *For all* $\mathfrak{C} \in \mathbb{C}$ *there is* $\mathfrak{D} \in \mathbb{D}$ *such that* $\mathfrak{C} \subseteq \mathfrak{D}$.

References

[1] Shelah S. On the Arrow property. Adv. in Ap. Mat., vol. 34 (2005), pp. 217–251.

[2] Aleskerov F. T. Arrovian Aggregation Models. Springer US (1999).

[3] Polyakov N., Shamolin M. On a generalization of Arrow's impossibility theorem. Doklady Mathematics, vol. 89, no. 3 (2014), pp. 290-292.

[4] Polyakov, N. Galois connections for classes of discrete functions and their application to mathematical problems of social choice theory. PhD thesis. Moscow University (2016), Russian.

[5] Arrow K. Social Choice and Individual Values. 2 edition. Yale University Press (1963).

[6] Moulin, H. Axioms of Cooperative Decision Making. Cambridge University Press (1991).