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Tori in the Cremona groups

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Abstract. We classify up to conjugacy all subgroups of certain types in the full, affine and special affine Cremona groups and prove that the normalizers of these subgroups are algebraic. As an application, we obtain new results on the linearization problem by generalizing Białynicki-Birula's results of 1966–67 to disconnected groups. We prove fusion theorems for n-dimensional tori in the affine and special affine Cremona groups of rank n, and introduce and discuss the notions of Jordan decomposition and torsion primes for the Cremona groups.

Keywords: Cremona group, affine Cremona group, algebraic torus, diagonalizable algebraic group, conjugate subgroups, fusion theorems, torsion primes.

To I. R. Shafarevich on his 90th birthday

§ 1. Introduction

This work arose from an attempt to solve a problem posed in [1], [2], where we introduced the notions of a root α and a root vector δ of an affine algebraic variety X with respect to an algebraic torus $T \subseteq \operatorname{Aut} X$. Namely, δ is a locally nilpotent derivation of the coordinate algebra of X, and α is a character of T such that $t^* \circ \delta \circ (t^*)^{-1} = \alpha(t)\delta$ for all $t \in T$. These definitions are inspired by a natural analogy with the classical definitions of the theory of algebraic groups. They mean an attempt to apply to the (generally infinite-dimensional) group $\operatorname{Aut} X$ a technique that is important in the theory of ordinary algebraic groups.[1]

In [1], [2] we posed the following two problems in the classical case when $X = \mathbf{A}^n$ and $T = D_n^*$ is a maximal diagonal torus preserving the standard volume form (see formula (9) below).

(R) Find all roots and root vectors of the variety $\mathbf{A}^n$ with respect to D_n^*.

(W) Describe the normalizer and centralizer of the torus D_n^* in the group $\operatorname{Aut}^* \mathbf{A}^n$ of volume-preserving automorphisms of $\mathbf{A}^n$.

[1] In [1], [2] we considered the case when $X = \mathbf{A}^n$ and T is the maximal diagonal torus that preserves the standard volume form. However, this restriction plays no role in the definitions of a root and a root vector.

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Problem (R) was solved by Liendo [3], who obtained the following result. Let $x_1,\dots,x_n$ be the standard coordinate functions on $\mathbf{A}^n$, and let $\varepsilon_1,\dots,\varepsilon_n$ be the 'coordinate' characters of the standard n-dimensional diagonal torus D_n in $\operatorname{Aut}\mathbf{A}^n$ (see formulae (7) and (10) below). Then, up to multiplication by a non-zero constant, the root vectors are precisely all derivations δ of the form

$$x_1^{l_1}\cdots x_n^{l_n}\frac{\partial}{\partial x_i}, \tag{1}$$

where $l_1,\dots,l_n$ are non-negative integers and $l_i=0$. The root α corresponding to the root vector (1) is the restriction to D_n^* of the character

$$\varepsilon_i^{-1}\prod_{j=1}^{n}\varepsilon_j^{l_j}.$$

The problem mentioned at the beginning of this introduction is problem (W). Clearly, it is aimed at getting a description of the 'Weyl group' of the root system in problem (R). We solve it in the present paper. Namely, we prove (Theorems 6(ii)(a) and 12) that the normalizer (centralizer) of the torus D_n^* in $\operatorname{Aut}^*\mathbf{A}^n$ coincides with its normalizer (centralizer) in SL_n, whence the Weyl group of D_n^* in $\operatorname{Aut}^*\mathbf{A}^n$ is the same as that of D_n^* in SL_n: it is the group of all permutations of the characters $\varepsilon_1,\dots,\varepsilon_n$.

This result is in fact only a special case in a series of general results that we obtain here. Namely, D_n^* is only one of the infinitely many non-conjugate diagonalizable algebraic subgroups G of dimension $\geqslant n-1$ in the group $\operatorname{Aut}\mathbf{A}^n$. We shall prove that the normalizer of G in $\operatorname{Aut}\mathbf{A}^n$ is always an *algebraic* subgroup in $\operatorname{Aut}\mathbf{A}^n$ (Theorem 14). This property is characteristic for the dimensions specified: it does not generally hold for diagonalizable subgroups of dimension $\leqslant n-2$. In the case of the existence of non-constant G-invariant polynomial functions on $\mathbf{A}^n$, we explicitly describe the normalizer of G in $\operatorname{Aut}\mathbf{A}^n$. In particular, we show that in all cases but one it coincides with the normalizer of G in a group conjugate to GL_n (Theorem 6).

Using this information, we obtain new results on the linearization problem. In 1966–67 Białynicki-Birula [4], [5] proved that every algebraic action on $\mathbf{A}^n$ of an algebraic torus of dimension $\geqslant n-1$ is equivalent to a linear action. We extend this assertion to disconnected groups by proving that every algebraic action on $\mathbf{A}^n$ of either an n-dimensional algebraic group whose connected component of identity is a torus or an $(n-1)$-dimensional diagonalizable group is equivalent to a linear action (Theorems 11, 13).

We also obtain the following complete classifications.

(i) Classification of the diagonalizable subgroups in the group Aff_n of affine transformations (see (6) below) up to conjugacy in the full Cremona group $\mathrm{Cr}_n=\operatorname{Bir}\mathbf{A}^n$ (Theorem 1).

(ii) Classification of the n-dimensional diagonalizable subgroups of $\operatorname{Aut}\mathbf{A}^n$ up to conjugacy in $\operatorname{Aut}\mathbf{A}^n$ (Theorem 9).

(iii) Classification of the $(n-1)$-dimensional diagonalizable subgroups of $\operatorname{Aut}^*\mathbf{A}^n$ up to conjugacy in $\operatorname{Aut}^*\mathbf{A}^n$ (Theorem 10).

(iv) Classification up to conjugacy in $\mathrm{Aut}^*\mathbf{A}^n$ (resp. $\mathrm{Aut}\,\mathbf{A}^n$) of the maximal $(n-1)$-dimensional (resp. n-dimensional) algebraic subgroups G in $\mathrm{Aut}^*\mathbf{A}^n$ (resp. $\mathrm{Aut}\,\mathbf{A}^n$) such that G^0 is a torus (Theorems 12 and 11 respectively).

(v) Classification of the $(n-1)$-dimensional diagonalizable subgroups of $\mathrm{Aut}\,\mathbf{A}^n$ up to conjugacy in $\mathrm{Aut}\,\mathbf{A}^n$ (Theorem 13).

(vi) Classification of the diagonalizable subgroups of dimension $\geqslant n-1$ in $\mathrm{Aut}\,\mathbf{A}^n$ up to conjugacy in Cr_n (Theorems 9, 15).

(vii) Classification of the one-dimensional tori in $\mathrm{Aut}\,\mathbf{A}^3$ up to conjugacy in $\mathrm{Aut}\,\mathbf{A}^3$ (Theorem 16).

For example, we prove that the set of conjugacy classes of diagonalizable $(n-1)$-dimensional subgroups of $\mathrm{Aut}\,\mathbf{A}^n$ is bijectively parametrized by the set of all non-decreasing non-zero sequences

$$(l_1,\dots,l_n)\in\mathbb{Z}^n \tag{2}$$

such that $(l_1,\dots,l_n)\leqslant(-l_n,\dots,-l_1)$ in the lexicographic ordering. This parametrization associates the sequence (2) with the conjugacy class of the subgroup $\ker\varepsilon_1^{l_1}\cdots\varepsilon_n^{l_n}$.

As another example, we show that the diagonalizable subgroups of Aff_n are conjugate in Cr_n if and only if they are isomorphic. We also specify canonical representatives of them. In particular (see Corollary 5), any two isomorphic finite Abelian subgroups of Aff_n are conjugate in Cr_n (for finite cyclic subgroups, this was proved in [6]).

Serre [7] proved a fusion theorem for the torus D_n in Cr_n. We shall prove and use fusion theorems for n-dimensional tori in $\mathrm{Aut}\,\mathbf{A}^n$ and $\mathrm{Aut}^*\mathbf{A}^{n+1}$ (Theorem 7).

In the final section, developing the theme of the analogy between the Cremona groups and algebraic groups, we introduce and discuss the notions of Jordan decomposition and torsion primes for the Cremona groups. In the course of the discussion, we state some open problems.

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§ 2. Notation and conventions

In what follows, ‘variety’ means ‘algebraic variety over a fixed algebraically closed field k of characteristic zero’ in the sense of Serre [8].

Besides the standard notation and conventions in [9], [10], which will be used without special reference, we also adopt the following: $\mathrm{Mat}_{m\times n}(R)$ is the set of all matrices with m rows, n columns and entries in R; $N_H(S)$ (resp. $Z_H(S)$) is the normalizer (resp. centralizer) of a subgroup S in a group H; μ_d is the subgroup of order d in $\mathbf{G}_{\mathrm{m}}$; $\mathrm{X}(D)$ is the group of rational characters of a diagonalizable algebraic group D; $\chi(X)$ is the Euler characteristic of a variety X with respect to the l-adic cohomology (when $k=\mathbb{C}$, it is equal by [11] to the Euler characteristic with respect to ordinary cohomology with compact supports; see also [12], Appendix).

Let $\varphi\colon G\times M\to M$ be an action of a group G on a set M. Given any subsets $S\subseteq G$ and $X\subseteq M$, we denote the subset $\varphi(S\times X)\subseteq M$ by $S\cdot X$ (it will always be clear from the context which φ is meant). In particular, the G-orbit of a point a

is denoted by $G \cdot a$. The G-stabilizer of a point a is denoted by G_a. We also write $x_1, \ldots, x_n$ for the standard coordinate functions on $\mathbf{A}^n$:

$$x_i(a) := a_i, \qquad a := (a_1, \ldots, a_n) \in \mathbf{A}^n.$$

All algebraic groups are assumed to be affine, and all homomorphisms of such groups are assumed to be algebraic. All tori and diagonalizable groups are also assumed to be algebraic.

An action of a group G on a vector space V is said to be *locally finite* if, for every vector $v \in V$, the linear span of the orbit $G \cdot v$ is finite-dimensional.

The group $\mathrm{Cr}_n := \mathrm{Bir}\,\mathbf{A}^n$ is called the *Cremona group of rank n*. The map $\varphi \mapsto (\varphi^*)^{-1}$ identifies it with $\mathrm{Aut}_k\, k(x_1, \ldots, x_n)$, and every birational isomorphism $X \dashrightarrow \mathbf{A}^n$ identifies it with $\mathrm{Bir}\, X$. For every $g \in \mathrm{Cr}_n$, the functions

$$g_i = g^*(x_i) \in k(\mathbf{A}^n) \tag{3}$$

determine g by the formula

$$g(a) = (g_1(a), \ldots, g_n(a)) \quad \text{if } g \text{ is defined at } a \in \mathbf{A}^n, \tag{4}$$

and we write

$$(g_1, \ldots, g_n) := g. \tag{5}$$

Using the notion of an 'algebraic family' $S \to \mathrm{Cr}_n$ (see [13]), the group Cr_n is endowed with the Zariski topology (see [7], [14]). If a homomorphism $G \to \mathrm{Cr}_n$ of an algebraic group G is an algebraic family, then its image is called an *algebraic subgroup* of Cr_n (see [15]).

The *affine Cremona group of rank n* is the following subgroup $\mathrm{Aut}\,\mathbf{A}^n$ of Cr_n:

$$\mathrm{Aut}\,\mathbf{A}^n := \bigl\{(g_1, \ldots, g_n) \in \mathrm{Cr}_n \mid g_1, \ldots, g_n \in k[\mathbf{A}^n] = k[x_1, \ldots, x_n]\bigr\}.$$

It contains the algebraic subgroup of affine transformations

$$\mathrm{Aff}_n := \bigl\{(g_1, \ldots, g_n) \in \mathrm{Aut}\,\mathbf{A}^n \mid \deg g_1 = \cdots = \deg g_n = 1\bigr\}, \tag{6}$$

and Aff_n contains the algebraic subgroup of linear transformations

$$\mathrm{GL}_n := \{g \in \mathrm{Aff}_n \mid g(0) = 0\}.$$

If $g = (g_1, \ldots, g_n) \in \mathrm{Aut}\,\mathbf{A}^n$ (see (5)), then we put

$$\mathrm{Jac}(g) := \det\Bigl(\frac{\partial g_i}{\partial x_j}\Bigr).$$

Since $g \in \mathrm{Aut}\,\mathbf{A}^n$, we have $\mathrm{Jac}(g) \in k \setminus \{0\}$. Therefore $g \mapsto \mathrm{Jac}(g)$ is a homomorphism of $\mathrm{Aut}\,\mathbf{A}^n$ into the multiplicative group of the field k. Its kernel

$$\mathrm{Aut}^*\mathbf{A}^n := \{g \in \mathrm{Aut}\,\mathbf{A}^n \mid \mathrm{Jac}(g) = 1\}$$

consists of those automorphisms of $\mathbf{A}^n$ that preserve the standard volume form and is called the *special affine Cremona group of rank* $n-1$ (concerning the ranks in this terminology, see Theorems 2(i) and 4(i)). The last group contains an algebraic subgroup

$$\mathrm{SL}_n := \mathrm{GL}_n \cap \mathrm{Aut}^* \mathbf{A}^n.$$

The embeddings $\mathrm{Cr}_n \hookrightarrow \mathrm{Cr}_{n+1}$, $(g_1, \dots, g_n) \mapsto (g_1, \dots, g_n, x_{n+1})$, form a tower $\mathrm{Cr}_1 \hookrightarrow \mathrm{Cr}_2 \hookrightarrow \cdots \hookrightarrow \mathrm{Cr}_n \hookrightarrow \cdots$. Its direct limit Cr_∞ is called the *Cremona group of infinite rank* (see [15], Section 1).

In GL_n we distinguish the 'standard' maximal torus

$$D_n := \{(t_1 x_1, \dots, t_n x_n) \mid t_1, \dots, t_n \in k\} \subset \mathrm{GL}_n\,. \tag{7}$$

Its normalizer in GL_n is the group of all monomial transformations in GL_n:

$$N_{\mathrm{GL}_n}(D_n) := \bigl\{(t_1 x_{\sigma(1)}, \dots, t_n x_{\sigma(n)}) \mid \sigma \in S_n,\ t_1, \dots, t_n \in k\bigr\} \subset \mathrm{GL}_n, \tag{8}$$

where S_n is the symmetric group of degree n. The group $\mathrm{Aut}^* \mathbf{A}^n$ contains the torus

$$D_n^* := D_n \cap \mathrm{Aut}^* \mathbf{A}^n = \bigl\{(t_1 x_1, \dots, t_n x_n) \mid t_1, \dots, t_n \in k,\ t_1 \cdots t_n = 1\bigr\}. \tag{9}$$

The 'coordinate' characters $\varepsilon_1, \dots, \varepsilon_n$ of the torus D_n are given by

$$\varepsilon_i \colon D_n \to \mathbf{G}_{\mathrm{m}}, \qquad (t_1 x_1, \dots, t_n x_n) \mapsto t_i. \tag{10}$$

They form a basis of the (free Abelian) group $\mathrm{X}(D_n)$.

§ 3. Some subgroups of $\mathbf{Cr}_n$

In what follows we regard elements of $\mathbb{Z}^n$ as row-vectors of length n with integer components. Then the rows of any matrix $A = (a_{ij}) \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$ are elements of this group. We shall use the notation

$$\mathcal{R}_A := \text{ the subgroup of } \mathbb{Z}^n \text{ generated by the rows of } A, \tag{11}$$

$$D_n(A) = \bigcap_{i=1}^{m} \ker \lambda_i, \quad \text{where} \quad \lambda_i := \varepsilon_1^{a_{i,1}} \cdots \varepsilon_n^{a_{i,n}}. \tag{12}$$

When $m = 1$, we write $D_n(l_1, \dots, l_n)$ instead of $D_n((l_1 \ \dots \ l_n))$. In particular,

$$D_n(0, \dots, 0) = D_n, \qquad D_n(1, \dots, 1) = D_n^*. \tag{13}$$

Clearly, $D_n(A)$ is a closed subgroup of D_n, and (see (11))

$$D_n(A) = \bigcap_{(l_1, \dots, l_n) \in \mathcal{R}(A)} \ker \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}. \tag{14}$$

We recall some terminology to be used below (see, for example, [16] and [17]).

Every finite Abelian group G can be decomposed into a direct sum of cyclic groups of orders $d_1, \dots, d_m$, where d_i divides d_{i+1} for $i = 1, \dots, m-1$, and $d_1 > 1$

if $|G| > 1$. The numbers $d_1, \ldots, d_s$ are uniquely determined by G and are called the *invariant factors of* G.

Every non-zero integer matrix A can be transformed by elementary transformations of rows and columns to a matrix $S = (s_{ij})$ whose only non-zero entries are s_{ii} for $i = 1, \ldots, r$ and s_{ii} divides $s_{i+1,i+1}$ for $i = 1, \ldots, r-1$. The integers $s_{11}, \ldots, s_{rr}$ are uniquely determined by A ($s_{ii} = f_i/f_{i-1}$, where f_i is the greatest common divisor of all the minors of order i of A and $f_0 := 1$) and are called the *invariant factors of* A. The matrix S is called the *Smith normal form* of A.

Lemma 1. *If B is obtained from $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ by elementary transformations of rows and columns, then the subgroups $D_n(A)$ and $D_n(B)$ are conjugate in Cr_n.*

Proof. Let $\tau_1, \ldots, \tau_n$ be a basis of the group $\mathrm{X}(D_n)$. Then (see (12)) we have $\lambda_i = \tau_1^{c_{i1}} \cdots \tau_n^{c_{in}}$ for some $c_{ij} \in \mathbb{Z}$ and

$$D_n(A) = \bigcap_{i=1}^{m} \ker \tau_1^{c_{i1}} \cdots \tau_n^{c_{in}}. \tag{15}$$

The group $\mathrm{Aut}_{\mathrm{gr}} D_n$ of automorphisms of the algebraic group D_n is naturally identified with $\mathrm{GL}_n(\mathbb{Z})$. Its natural action on the set of bases of the group $\mathrm{X}(D_n)$ is transitive. Hence there is an automorphism

$$\varphi \in \mathrm{Aut}_{\mathrm{gr}} D_n \tag{16}$$

such that $\tau_i \circ \varphi = \varepsilon_i$ for all i. Then it follows from (15) that

$$\varphi^{-1}(D_n(A)) = D_n(C), \quad \text{where} \;\; C = (c_{ij}) \in \mathrm{Mat}_{m\times n}(\mathbb{Z}). \tag{17}$$

Since the map of varieties $D_n \to \mathbf{A}^n$, $(t_1x_1, \ldots, t_nx_n) \mapsto (t_1, \ldots, t_n)$, is a birational isomorphism, we can use it to identify the group $\mathrm{Cr}_n = \mathrm{Bir}\,\mathbf{A}^n$ with the group of birational automorphisms of the underlying variety of the torus D_n. Then φ becomes an element of Cr_n, and we easily deduce from (16) and (17) that the following equality holds in this group:

$$\varphi^{-1} D_n(A) \varphi = D_n(C).$$

Note that if a basis $\tau_1, \ldots, \tau_n$ is obtained from the basis $\varepsilon_1, \ldots, \varepsilon_n$ by an elementary transformation, then the matrix C is obtained from A by an elementary transformation of columns, and every elementary transformation of columns of A is realizable in this way. Moreover, if the sequence $\varphi_1, \ldots, \varphi_m \in \mathrm{X}(D_n)$ is obtained by an elementary transformation from the sequence $\lambda_1, \ldots, \lambda_m$, then $D_n(A) = \bigcap_{i=1}^m \ker \varphi_i$, the matrix (c_{ij}) defined by the equalities $\varphi_i = \varepsilon_1^{c_{i1}} \cdots \varepsilon_n^{c_{in}}$ is obtained from A by an elementary transformation of rows, and every elementary transformation of rows of A is realizable in this way. Clearly, this proves the lemma. □

Corollary 1. *If S is the Smith normal form of A, then the subgroups $D_n(A)$ and $D_n(S)$ are conjugate in Cr_n.*

Lemma 2. (i) *If $q_1 \leqslant \dots \leqslant q_r$ are the invariant factors of a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$, then the group $D_n(A)$ is isomorphic to*

$$\boldsymbol{\mu}_{q_1} \times \dots \times \boldsymbol{\mu}_{q_r} \times \mathbf{G}_{\mathrm{m}}^{n-r}. \tag{18}$$

(ii) *The closed $(n-m)$-dimensional subgroups of D_n are nothing but all possible subgroups $D_n(A)$, where $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and $\operatorname{rk} A = m$.*

(iii) $\mathcal{R}_A = \{(l_1, \dots, l_n) \in \mathbb{Z}^n \mid D_n(A) \subseteq \ker \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}\}$ *for every matrix* $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$.

(iv) *If $A \in \mathrm{Mat}_{s\times n}(\mathbb{Z})$ and $B \in \mathrm{Mat}_{t\times n}(\mathbb{Z})$, then the following assertions hold.*

(a) *$D_n(A) = D_n(B)$ if and only if $\mathcal{R}_A = \mathcal{R}_B$.*

(b) *The following properties are equivalent*:

(b_1) *$D_n(A)$ and $D_n(B)$ are conjugate in GL_n*;

(b_2) *$D_n(A)$ and $D_n(B)$ are conjugate in $N_{\mathrm{GL}_n}(D_n)$*;

(b_3) *there is a permutation of columns that transforms B into a matrix C such that $\mathcal{R}_A = \mathcal{R}_C$.*

Proof. (i) Let $S = (s_{ij})$ be the Smith normal form of A. Then $s_{11} = q_1, \dots, s_{rr} = q_r$, and $s_{ij} = 0$ otherwise. Hence $D_n(S)$ is isomorphic to the group (18). But $D_n(A)$ is isomorphic to $D_n(S)$ by Corollary 1.

(ii) It follows from (i) that

$$\dim D_n(A) = n - \operatorname{rk} A, \tag{19}$$

whence $\dim D_n(A) = n - m$ when $\operatorname{rk} A = m$. Conversely, let H be a closed subgroup of D_n with $\dim H = n - m$. Then D_n/H is an m-dimensional torus ([9], p. 114) and, therefore, there is an isomorphism $\alpha\colon D_n/H \to \mathbf{G}_{\mathrm{m}}^m$. Let $\lambda_i \in \mathrm{X}(D_n)$ be the composite of the homomorphisms

$$D_n \xrightarrow{\pi} D_n/H \xrightarrow{\alpha} \mathbf{G}_{\mathrm{m}}^m \xrightarrow{\mathrm{pr}_i} \mathbf{G}_{\mathrm{m}},$$

where π is the canonical projection and pr_i is the projection to the ith factor. Then $H = \bigcap_{i=1}^m \ker \lambda_i$. We now deduce from (12) that $H = D_n(A)$ and from (19) that $\operatorname{rk} A = m$.

(iii) It follows from (11), (12) that the left-hand side of the equality to be proved is contained in the right-hand side. To prove the opposite inclusion, we consider a character $\lambda = \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}$ whose kernel contains $D_n(A)$. Without changing $D_n(A)$ and $\mathcal{R}_A$, we can leave in A only rows that form a base of the group $\mathcal{R}_A$ and remove the others. This reduces us to the case when $\operatorname{rk} A = m$. Then we consider the characters $\lambda_1, \dots, \lambda_m$ as defined in (12) and the homomorphism

$$\varphi\colon D_n \to \mathbf{G}_{\mathrm{m}}^m, \quad g \mapsto (\lambda_1(g), \dots, \lambda_m(g)).$$

By (12) we have $\ker \varphi = D_n(A)$. It follows from this and (19) that $\dim \varphi(D_n) = m$. Therefore φ is a surjection. Hence $\mathbf{G}_{\mathrm{m}}^m$ is the quotient of D_n by $D_n(A)$, and φ is the canonical homomorphism onto this quotient. Since the character λ is constant on the fibres of φ, the universal property of quotients implies that there is a character

$\mu\colon \mathbf{G}_{\mathrm{m}}^m \to \mathbf{G}_{\mathrm{m}}$ such that $\lambda = \mu \circ \varphi$. Hence $\lambda = \lambda_1^{c_1} \cdots \lambda_m^{c_m}$ for some $c_1, \ldots, c_m \in \mathbb{Z}$. This means that $(l_1, \ldots, l_n) \in \mathcal{R}_A$.

(iv)(a) If $D_n(A) = D_n(B)$, then $\mathcal{R}_A = \mathcal{R}_B$ because of (iii). Conversely, if $\mathcal{R}_A = \mathcal{R}_B$, then $D_n(A) = D_n(B)$ because of (14).

(iv)(b) By the fusion theorem ([18], § 1.1.1), the subgroups $D_n(A)$ and $D_n(B)$ are conjugate in GL_n if and only if they are conjugate in $N_{\mathrm{GL}_n}(D_n)$. But (8) and (12) imply that $D_n(A)$ and $D_n(B)$ are conjugate in $N_{\mathrm{GL}_n}(D_n)$ if and only if one can transform B by a permutation of columns to a matrix C such that $D_n(A) = D_n(C)$. By (iii)(a), this equality is equivalent to the equality $\mathcal{R}_C = \mathcal{R}_A$. □

Corollary 2. (i) *If* $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$ *and* $d := \mathrm{GCD}(l_1, \ldots, l_n)$, *then* $D_n(l_1, \ldots, l_n)$ *is isomorphic to* $\mu_d \times \mathbf{G}_{\mathrm{m}}^{n-1}$. *In particular,* $D_n(l_1, \ldots, l_n)$ *is connected* (*and hence is a torus*) *if and only if* $d = 1$.

(ii) *The closed* $(n-1)$*-dimensional subgroups of* D_n *are nothing but all possible subgroups* $D_n(l_1, \ldots, l_n)$ *with* $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$.

(iii) $D_n(l_1, \ldots, l_n) = D_n(l'_1, \ldots, l'_n)$ *if and only if*

$$(l_1, \ldots, l_n) = \pm(l'_1, \ldots, l'_n).$$

(iv) *The following properties are equivalent.*

(iv$_1$) $D_n(l_1, \ldots, l_n)$ *and* $D_n(l'_1, \ldots, l'_n)$ *are conjugate in* GL_n;

(iv$_2$) $D_n(l_1, \ldots, l_n)$ *and* $D_n(l'_1, \ldots, l'_n)$ *are conjugate in* $N_{\mathrm{GL}_n}(D_n)$;

(iv$_3$) *there is a permutation* $\sigma \in S_n$ *such that*

$$(l_1, \ldots, l_n) = \pm(l'_{\sigma(1)}, \ldots, l'_{\sigma(n)}).$$

The following lemma gives an effective numerical criterion for the equality $\mathcal{R}_A = \mathcal{R}_B$ in Lemma 2(iii).

Suppose that $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$, $\mathrm{rk}\, A = m$. For every strictly increasing m-tuple of integers $i_1, \ldots, i_m$ belonging to the interval $[1, n]$ we put

$$p_{i_1,\ldots,i_m}(A) := \det A_{i_1,\ldots,i_m}, \tag{20}$$

where $A_{i_1,\ldots,i_m}$ is the submatrix of A formed by intersecting the rows with numbers $1, \ldots, m$ and the columns with numbers $i_1, \ldots, i_m$ (it is natural to call the $p_{i_1,\ldots,i_m}(A)$ the *Plücker coordinates of* A).

Lemma 3. *For any two matrices* $A, B \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ *of rank* m, *the following properties are equivalent.*

(i) $\mathcal{R}_A = \mathcal{R}_B$.

(ii) *Two conditions hold*:

(a) *either* $p_{i_1,\ldots,i_m}(A) = p_{i_1,\ldots,i_m}(B)$ *for all* $i_1, \ldots, i_m$, *or* $p_{i_1,\ldots,i_m}(A) = -\,p_{i_1,\ldots,i_m}(B)$ *for all* $i_1, \ldots, i_m$;

(b) *for every sequence* $i_1, \ldots, i_m$ *with* $p_{i_1,\ldots,i_m}(A) \neq 0$, *we have*

$$B_{i_1,\ldots,i_m}(A_{i_1,\ldots,i_m})^{-1} \in \mathrm{Mat}_{m\times m}(\mathbb{Z}). \tag{21}$$

Proof. Since $\operatorname{rk} A = \operatorname{rk} B = m$, the rows of A and B form bases in $\mathcal{R}_A$ and $\mathcal{R}_B$ respectively. Therefore $\mathcal{R}_A = \mathcal{R}_B$ if and only if there is a matrix $Q \in \mathrm{GL}_m(\mathbb{Z})$ such that

$$A = QB. \tag{22}$$

Suppose that $\mathcal{R}_A = \mathcal{R}_B$. Then (22) implies that

$$A_{i_1,\dots,i_m} = QB_{i_1,\dots,i_m} \tag{23}$$

for all $i_1,\dots,i_m$ and, therefore, $p_{i_1,\dots,i_m}(A) = \det Q p_{i_1,\dots,i_m}(B)$ by (20). Since $Q \in \mathrm{GL}_m(\mathbb{Z})$, we have $\det Q = \pm 1$. Hence condition (ii)(a) holds. If $p_{i_1,\dots,i_m}(A) \neq 0$, then we also have $p_{i_1,\dots,i_m}(B) \neq 0$, whence $B_{i_1,\dots,i_m}$ is non-singular and (23) says that $Q = A_{i_1,\dots,i_m}(B_{i_1,\dots,i_m})^{-1}$. Hence condition (ii)(b) holds. This proves the implication (i)⇒(ii).

To prove the reverse implication, we regard $\mathbb{Z}^n$ as a subset of the coordinate vector space (of rows) $\mathbb{Q}^n$. Condition (ii)(a) shows that the $\mathbb{Q}$-linear spans of the subsets $\mathcal{R}_A$ and $\mathcal{R}_B$ in $\mathbb{Q}^n$ have the same Plücker coordinates. Hence these spans are equal to the same vector subspace L (see, for example, [19], Theorem 10.1). Since the rows of A and the rows of B form two bases of L, there is a matrix $P \in \mathrm{GL}_m(\mathbb{Q})$ such that $A = PB$. Hence $A_{i_1,\dots,i_m} = PB_{i_1,\dots,i_m}$ for all $i_1,\dots,i_m$ and, therefore, $P = A_{i_1,\dots,i_m}(B_{i_1,\dots,i_m})^{-1}$ if $p_{i_1,\dots,i_m}(B) \neq 0$. It then follows from (ii)(b) that $P \in \mathrm{GL}_n(\mathbb{Z})$. Therefore $\mathcal{R}_A = \mathcal{R}_B$. This proves the implication (ii)⇒(i). □

Remark 1. The proof of Lemma 3 shows that (ii)(a) actually implies that the matrix $B_{i_1,\dots,i_m}(A_{i_1,\dots,i_m})^{-1}$ is independent of the choice of a sequence $i_1,\dots,i_m$ with $p_{i_1,\dots,i_m}(A) \neq 0$. Therefore (ii)(b) follows from (ii)(a) provided that (21) holds for at least *one* such sequence.

Remark 2. If $m = 1$, then condition (ii)(b) of Lemma 3 follows from (ii)(a) (but this is not the case for $m > 1$).

Theorem 1 (classification of diagonalizable subgroups of Aff_n up to conjugacy in Cr_n). (i) *Two diagonalizable subgroups of* Aff_n *are conjugate in* Cr_n *if and only if they are isomorphic.*

(ii) *Any diagonalizable subgroup G of* Aff_n *is conjugate in* Cr_n *to a unique closed subgroup of D_n of the form*

$$\ker \varepsilon_{r+1}^{d_1} \cap \dots \cap \ker \varepsilon_{r+s}^{d_s} \cap \ker \varepsilon_{r+s+1} \cap \dots \cap \ker \varepsilon_n, \tag{24}$$

where $0 \leqslant r \leqslant n$, $0 \leqslant s \leqslant n$, $r+s \leqslant n$, $2 \leqslant d_1$ and d_i divides d_{i+1} for every $i < s$. The integers that determine the subgroup (24) *have the following meaning*: $r = \dim G$, *and $d_1,\dots,d_s$ are the invariant factors of the finite Abelian group G/G^0.*

Proof. Since all maximal reductive subgroups of an algebraic group are conjugate (see [20], § 5.1) and GL_n is one of them in Aff_n, we see that every diagonalizable subgroup of Aff_n is conjugate to a subgroup of GL_n. In its turn, every diagonalizable subgroup of GL_n is conjugate to a subgroup of the torus D_n (see [9], Ch. I, § 4.6). In view of Lemma 2(ii), this shows that it suffices to prove (i) for the subgroups $D_n(A)$ and $D_n(B)$ of the torus D_n. Adding zero rows if necessary, we may assume that A and B have the same number of rows. Suppose that the groups $D_n(A)$ and $D_n(B)$ are isomorphic. Then their dimensions are equal, and the groups $D_n(A)/D_n(A)^0$ and $D_n(B)/D_n(B)^0$ have the same invariant factors. This and Lemma 2(i) yield that the matrices A and B have the same invariant factors (which are obtained by adding equal numbers of ones to the invariant factors of the previous groups). Hence the Smith normal forms of A and B coincide. By Corollary 1, it follows that $D_n(A)$ and $D_n(B)$ are conjugate in Cr_n. This proves (i).

Clearly, the integers that determine the subgroup (24) have the meaning specified in (ii). Since every diagonalizable group is a direct product of a finite Abelian group and a torus, it is uniquely determined (up to isomorphism) by its dimension and the invariant factors of the group of connected components. This and (i) yield (ii). □

Corollary 3. *The subgroups $D_n(A)$ and $D_n(B)$ are conjugate in Cr_n if and only if A and B have the same invariant factors.*

Corollary 4. *Every torus T in Aff_n is conjugate in Cr_n to the torus D_r, where $r = \dim T$.*

Proof. This follows from Theorem 1(ii). □

Corollary 5. *Any two isomorphic finite Abelian subgroups of Aff_n are conjugate in Cr_n.*

Proof. Since $\operatorname{char} k = 0$, every element of finite order in Aff_n is semisimple. Therefore every finite Abelian subgroup of Aff_n is reductive and hence conjugate in Aff_n to a subgroup of GL_n (see the proof of Theorem 1). But every commutative subgroup of GL_n containing only semisimple elements is diagonalizable (see [9], Proposition 4.6(b)). The desired assertion now follows from Theorem 1 (i).

Corollary 6 ([6], Theorem 1). *Any two elements of the same finite order in Aff_n are conjugate in Cr_n.*

§ 4. Tori in $\mathbf{Cr}_n$, $\mathbf{Aut\,A}^n$ and $\mathbf{Aut^*\,A}^n$

Theorem 2 (tori in Cr_n). (i) Cr_n *contains no tori of dimension $> n$.*

(ii) *Every r-dimensional torus in Cr_n for $r = n, n-1, n-2$ is conjugate to the torus D_r.*

(iii) *If $n \geqslant 5$, then there are $(n-3)$-dimensional tori in Cr_n that are not conjugate to subtori of D_n.*

(iv) *Every r-dimensional torus in Cr_n is conjugate in Cr_{n+r} to the torus D_r.*

(v) *Every r-dimensional torus in Cr_∞ is conjugate to the torus D_r.*

Proof. Part (i) is proved, for example, in [21] (see also [15], Corollary 2.2).

By [4], Corollary 2 (see also [15], Corollary 2.4(b)), every r-dimensional torus in Cr_n for $r = n, n-1, n-2$ is conjugate to a subtorus of D_n. Therefore part (ii) follows from Corollary 4.

Part (iii) is proved in [15], Corollary 2.5.

By [15], Theorem 2.6, every r-dimensional torus in Cr_n is conjugate in Cr_{n+r} to a subtorus of D_{n+r}. Therefore part (iv) follows from Corollary 4.

Part (v) follows from part (iv). □

Corollary 7. (i) *Every n-dimensional torus in* Cr_n *is maximal.*

(ii) *There are no maximal $(n-1)$- or $(n-2)$-dimensional tori in* Cr_n.

(iii) *For $n \geqslant 5$ there are maximal $(n-3)$-dimensional tori in* Cr_n.

Remark 3. For $n \leqslant 3$, Theorem 2 yields a classification of all tori in Cr_n up to conjugacy: the classes of conjugate non-trivial tori are exhausted by those of $D_1, \dots, D_n$.

Let $(l_1, \dots, l_n) \in \mathbb{Z}^n$ be a non-zero element with $\mathrm{GCD}(l_1, \dots, l_n) = 1$. We easily see that the homomorphism

$$\mathbf{G}_\mathrm{m} \to D_n, \qquad t \mapsto (t^{l_1}x_1, \dots, t^{l_n}x_n), \tag{25}$$

is an embedding, and every embedding $\mathbf{G}_\mathrm{m} \hookrightarrow D_n$ is of this form. We denote the image of the embedding (25) by $T(l_1, \dots, l_n)$. It is a one-dimensional torus in D_n, and all one-dimensional tori in D_n are of this form.

Lemma 4. *The following properties are equivalent*:

(i) $T(l_1, \dots, l_n) = T(l'_1, \dots, l'_n)$;

(ii) $(l_1, \dots, l_n) = \pm(l'_1, \dots, l'_n)$.

Proof. (i)⇒(ii). This is clear.

(ii)⇒(i). Suppose that (i) holds. Take an element $t \in \mathbf{G}_\mathrm{m}$ of infinite order. It follows from (i) and the definition of $T(l_1, \dots, l_n)$ that there is an element $s \in \mathbf{G}_\mathrm{m}$ such that $t^{l_i} = s^{l'_i}$ for all $i = 1, \dots, n$. Hence $t^{l_i l'_j} = s^{l'_i l'_j} = t^{l_j l'_i}$ for any distinct integers i, j in the interval $[1, n]$. Since the order of t is infinite, it follows that $l_i l'_j - l_j l'_i = 0$. Hence,

$$\mathrm{rk}\begin{pmatrix} l_1 & \dots & l_n \\ l'_1 & \dots & l'_n \end{pmatrix} = 1.$$

Therefore $(l_1, \dots, l_n) = \gamma(l'_1, \dots, l'_n)$ for some $\gamma \in \mathbb{Q}$ or, equivalently, $p(l_1, \dots, l_n) = q(l'_1, \dots, l'_n)$, where $p, q \in \mathbb{Z}$, $\mathrm{GCD}(p, q) = 1$. Hence p (resp. q) divides each of the integers $l'_1, \dots, l'_n$ (resp. $l_1, \dots, l_n$). Since the integers in each of these groups are coprime, we have $\gamma = \pm 1$, that is, (ii) holds. □

Theorem 3 (tori in $\mathrm{Aut}\,\mathbf{A}^n$). (i) *Every n-dimensional torus in* $\mathrm{Aut}\,\mathbf{A}^n$ *is conjugate to the torus* D_n.

(ii) *Up to conjugacy in* $\mathrm{Aut}\,\mathbf{A}^n$, *the $(n-1)$-dimensional tori in* $\mathrm{Aut}\,\mathbf{A}^n$ *are exhausted by the groups* $D_n(l_1, \dots, l_n)$, *where* $(l_1, \dots, l_n) \neq (0, \dots, 0)$ *and* $\mathrm{GCD}(l_1, \dots, l_n) = 1$.

(iii) *Up to conjugacy in* $\mathrm{Aut}\,\mathbf{A}^3$, *the one-dimensional tori in* $\mathrm{Aut}\,\mathbf{A}^3$ *are exhausted by the groups* $T(l_1, l_2, l_3)$.

Proof. By [4] (resp. [5]), every r-dimensional torus in $\mathrm{Aut}\,\mathbf{A}^n$ with $r = n$ (resp. $r = n-1$) is conjugate to a subtorus of D_n. This and Corollary 2 yield parts (i) and (ii).

By [22], every one-dimensional torus in $\mathrm{Aut}\,\mathbf{A}^3$ is conjugate to a subtorus of D_3. This proves (iii). □

Theorem 4 (tori in $\mathrm{Aut}^*\mathbf{A}^n$). (i) $\mathrm{Aut}^*\mathbf{A}^n$ *contains no tori of dimension* $> n-1$.

(ii) *Every $(n-1)$-dimensional torus in* $\mathrm{Aut}^*\mathbf{A}^n$ *is maximal and conjugate to* D_n^* (*see* (9)).

Proof. (i) If $\operatorname{Aut}^*\mathbf{A}^n$ contains an n-dimensional torus T, then Theorem 3(i) yields an element $g \in \operatorname{Aut}\mathbf{A}^n$ such that

$$T = gD_ng^{-1}. \tag{26}$$

Replacing g by gz, where $z \in D_n$ is any element with $\operatorname{Jac}(z) = \det z = 1/\operatorname{Jac}(g)$, we may assume that $g \in \operatorname{Aut}^*\mathbf{A}^n$. This and (26) imply that $D_n \subset \operatorname{Aut}^*\mathbf{A}^n$, a contradiction.

(ii) Let S be an $(n-1)$-dimensional torus in $\operatorname{Aut}^*\mathbf{A}^n$. By Theorem 3(i) and Corollary 2(ii) there are $g \in \operatorname{Aut}\mathbf{A}^n$ and $(l_1, \dots, l_n) \in \mathbb{Z}^n$ such that

$$S = gD_n(l_1, \dots, l_n)g^{-1}. \tag{27}$$

As in the proof of (i), we may assume that $g \in \operatorname{Aut}^*\mathbf{A}^n$. Then (27) yields that $D_n(l_1, \dots, l_n) \subset \operatorname{Aut}^*\mathbf{A}^n$. Since $D_n \cap \ker \operatorname{Jac} = D_n^* := D_n(1, \dots, 1)$, it follows that $D_n(l_1, \dots, l_n) \subseteq D_n^*$. By Corollary 2(i), both sides of this inclusion are $(n-1)$-dimensional tori. Hence the inclusion is an equality. □

Remark 4. In contrast to Cr_n (see Corollary 7(iii)), nothing is currently known regarding the existence of maximal tori of non-maximal dimension in $\operatorname{Aut}\mathbf{A}^n$ and $\operatorname{Aut}^*\mathbf{A}^n$. This problem is intimately related to the *cancellation problem*: is there an affine variety X non-isomorphic to $\mathbf{A}^m$, $m = \dim X$, such that $X \times \mathbf{A}^d$ is isomorphic to $\mathbf{A}^{m+d}$ for some d? If the answer is positive, then $\operatorname{Aut}\mathbf{A}^n$ and $\operatorname{Aut}^*\mathbf{A}^n$ for $n = m + d$ contain a maximal torus T of non-maximal dimension. Indeed, multiplying $X \times \mathbf{A}^d$ by $\mathbf{A}^1$ if necessary, we may assume that $d \geqslant 2$. Let λ be the character $t \mapsto t$ of the torus $\mathbf{G}_{\mathrm{m}}$. We consider a linear action of $\mathbf{G}_{\mathrm{m}}$ on $\mathbf{A}^d$ with precisely two isotypic components: $(d-1)$-dimensional of type λ and one-dimensional of type λ^{1-d}. It determines the action of $\mathbf{G}_{\mathrm{m}}$ on $X \times \mathbf{A}^d$ via the second factor and, therefore, an action of $\mathbf{G}_{\mathrm{m}}$ on $\mathbf{A}^n$. Consider the torus in $\operatorname{Aut}\mathbf{A}^n$ which is the image of $\mathbf{G}_{\mathrm{m}}$ under the homomorphism determined by this action. By construction, this torus lies in $\operatorname{Aut}^*\mathbf{A}^n$. Let T (resp. T') be a maximal torus in $\operatorname{Aut}\mathbf{A}^n$ (resp. $\operatorname{Aut}^*\mathbf{A}^n$) containing this image. If T is n-dimensional (resp. T' is $(n-1)$-dimensional), then it is conjugate to a subtorus of D_n by Theorem 3(i) (resp. Theorem 4) and, therefore, the action of T (resp. T') on $\mathbf{A}^n$ is equivalent to a linear action. Hence the action of $\mathbf{G}_{\mathrm{m}}$ on $\mathbf{A}^n$ is also equivalent to a linear action and, therefore, the set F of its fixed points is isomorphic to an affine space. However, by construction, there is only one $\mathbf{G}_{\mathrm{m}}$-fixed point in $\mathbf{A}^d$, whence F is isomorphic to X, a contradiction.[2]

§ 5. Orbits and stabilizers of the action of $D_n(l_1, \dots, l_n)$ on $\mathbf{A}^n$

Here we establish some properties (used in what follows) of orbits and stabilizers of the natural action on $\mathbf{A}^n$ of the group

$$G := D_n(l_1, \dots, l_n). \tag{28}$$

[2] *Added on August* 2, 2012. Theorems 3, 4 and Remark 4 remain valid in the case when the characteristic p of the ground field k is positive. The preprint [23] posted today asserts that the hypersurface X given by the equation $x_1^m x_2 + x_3^{p^e} + x_4 + x^{sp} = 0$ in $\mathbf{A}^4$, where m, e, s are positive integers with $p^e \nmid sp$ and $sp \nmid p^e$, is not isomorphic to $\mathbf{A}^3$, but $X \times \mathbf{A}^1$ is isomorphic to $\mathbf{A}^4$. By Remark 4, it follows that for $\operatorname{char} k > 0$ and every $n \geqslant 5$ there are maximal tori of non-maximal dimension in $\operatorname{Aut}\mathbf{A}^n$ and $\operatorname{Aut}^*\mathbf{A}^n$.

Clearly, every coordinate hypersurface

$$H_i := \{a \in \mathbf{A}^n \mid x_i(a) = 0\} \tag{29}$$

is G-invariant.

Lemma 5. *The G-stabilizer of every point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is trivial.*

(i) *If $(l_1, \dots, l_n) = (0, \dots, 0)$, then $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is a G-orbit, and the dimension of the G-stabilizer of every point of $\bigcup_{i=1}^n H_i$ is positive.*

(ii) *If $(l_1, \dots, l_n) \neq (0, \dots, 0)$, then $\dim G \cdot a = n - 1$ for every point $a \in \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$.*

Proof. This follows from (28), (12), (4) and Lemma 2. □

We now consider the case

$$(l_1, \dots, l_n) \neq (0, \dots, 0).$$

Lemma 6. *If $l_i \neq 0$, then the open subset $\mathcal{O}_i := H_i \setminus \bigcup_{j \neq i} H_j$ of H_i is a G-orbit.*

Proof. Since $\mathcal{O}_i$ is G-invariant, it suffices to prove that $\mathcal{O}_i$ is contained in a G-orbit. Since $l_i \neq 0$, the equation $x^{l_i} = \alpha$ has a solution for any $\alpha \in k$, $\alpha \neq 0$. This and (28), (12) imply that for any two points

$$b = (b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_n), c = (c_1, \dots, c_{i-1}, 0, c_{i+1}, \dots, c_n) \in H_i \setminus \bigcup_{j \neq i} H_j$$

there is an element $g = (t_1 x_1, \dots, t_n x_n) \in G$ such that $t_j = b_j^{-1} c_j$ for every $j \neq i$. We now obtain from (4) that $g \cdot b = c$, as required. □

Lemma 7. *The following properties are equivalent*:

(i) *the numbers $l_1, \dots, l_n$ are non-zero and have the same sign*;

(ii) *the G-orbit of any point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is closed in $\mathbf{A}^n$*;

(iii) *the G-orbit of some point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is closed in $\mathbf{A}^n$.*

Proof. Consider a point

$$a = (a_1, \dots, a_n) \in \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i.$$

Assume that (i) holds. Suppose that the G-orbit of a is not closed in $\mathbf{A}^n$. Then its boundary is accessible by a one-parameter subgroup, that is, one can find a homomorphism $\varphi\colon \mathbf{G}_{\mathrm{m}} \to G$ such that there is a limit

$$\lim_{t \to 0} \varphi(t) \cdot a = b \in \overline{G \cdot a} \setminus G \cdot a \tag{30}$$

(see [10], Theorem 6.9). Formula (30) means that the morphism $\mathbf{G}_{\mathrm{m}} = \mathbf{A}^1 \setminus \{0\} \to \mathbf{A}^n$, $t \mapsto \varphi(t) \cdot a$, extends to a morphism $\mathbf{A}^1 \to \mathbf{A}^n$ that maps 0 to the point b. Since φ is algebraic, there is a vector $(d_1, \dots, d_n) \in \mathbb{Z}^n$ such that $\varphi(t) = (t^{d_1} x_1, \dots, t^{d_n} x_n)$ for every $t \in \mathbf{G}_{\mathrm{m}}$. Since $\varphi(t) \cdot a = (t^{d_1} a_1, \dots, t^{d_n} a_n)$ and $a_i \neq 0$ for every i, the existence of the limit above means that

$$d_1 \geqslant 0, \ \dots, \ d_n \geqslant 0. \tag{31}$$

On the other hand, since $\varphi(t)\in G$, it follows from (28) and (12) that $t^{d_1l_1+\cdots+d_nl_n}=1$ for every t, that is,

$$d_1l_1+\cdots+d_nl_n=0. \tag{32}$$

But (31), (32) and condition (i) imply that $d_1=\cdots=d_n=0$. Hence $b=a$ contrary to (30). This contradiction proves that (i)⇒(ii).

For the converse, suppose that (i) does not hold, that is, the n-tuple $l_1,\dots,l_n$ contains either two non-zero numbers of different signs, or a zero. In the first case let, say, $l_1>0$, $l_2<0$. Then it follows from (12) that the image of the homomorphism $\varphi\colon \mathbf{G}_{\mathrm{m}}\to D_n,\ t\mapsto(t^{-l_2}x_1,t^{l_1}x_2,x_3,\dots,x_n)$, lies in G. Since

$$\lim_{t\to 0}\varphi(t)\cdot a=(0,0,a_3,\dots,a_n)\notin G\cdot a,$$

this shows that the orbit $G\cdot a$ is non-closed. In the second case we may assume that $a_1=0$. Then G contains the image of the homomorphism $\varphi\colon\mathbf{G}_{\mathrm{m}}\to D_n$, $t\mapsto(tx_1,x_2,\dots,x_n)$ and, since

$$\lim_{t\to 0}\varphi(t)\cdot a=(0,a_2,\dots,a_n)\notin G\cdot a,$$

the orbit $G\cdot a$ is non-closed. This proves the implication (iii)⇒(i). □

The following corollary is a consequence of Lemmas 5–7.

Corollary 8. *If $l_1,\dots,l_n$ are all non-zero and have the same sign, then $\mathbf{A}^n$ contains precisely n non-closed $(n-1)$-dimensional G-orbits: the orbits $\mathcal{O}_1,\dots,\mathcal{O}_n$ in Lemma 6.*

Remark 5. We recall from [24] that an action of an algebraic group on an algebraic variety is said to be *stable* if the orbits of points in general position are closed. Lemma 7 shows that the following properties are equivalent:

(i) $l_1,\dots,l_n$ are all non-zero and have the same sign;

(ii) the action of G on $\mathbf{A}^n$ is stable.

Lemma 8. *Assume that none of $l_1,\dots,l_n$ is equal to ±1. Then the following properties of a point $a=(a_1,\dots,a_n)\in\mathbf{A}^n$ are equivalent:*

(i) *a has a non-trivial G-stabilizer;*

(ii) *$a\in\bigcup_{i=1}^n H_i$.*

Proof. The implication (i)⇒(ii) follows from Lemma 5. We now assume that (ii) holds. Then there are $i_1,\dots,i_s$, $s\geqslant 1$, such that $a_j=0$ for $j=i_1,\dots,i_s$ and $a_j\neq 0$ for the other j. It follows from (4), (28) and (12) that an element $(t_1x_1,\dots,t_nx_n)\in D_n$ lies in the G-stabilizer of a if and only if $t_j=1$ for $j\neq i_1,\dots,i_s$ and

$$t_{i_1}^{l_{i_1}}\cdots t_{i_s}^{l_{i_s}}=1. \tag{33}$$

Since none of the numbers $l_1,\dots,l_n$ is equal to ±1 and k is an algebraically closed field of characteristic zero, it follows that (33) (regarded as the equation in $t_{i_1},\dots,t_{i_s}$) has at least two solutions. Hence the G-stabilizer of a is non-trivial. This proves that (ii)⇒(i). □

Lemma 9. *If the n-tuple $l_1, \dots, l_n$ contains two non-zero numbers of different signs, then the closure of every G-orbit contains* $(0, \dots, 0)$.

Proof. To be definite, assume that

$$l_1 > 0, \quad l_2 \geqslant 0, \ \dots, \ l_s \geqslant 0, \quad l_{s+1} < 0, \quad l_{s+2} \leqslant 0, \ \dots, \ l_n \leqslant 0.$$

Let d be a large positive integer such that

$$q := l_2 + \dots + l_s + dl_{s+1} + l_{s+2} + \dots + l_n < 0.$$

Since $-ql_1 + l_1l_2 + \dots + l_1l_s + l_1dl_{s+1} + l_1l_{s+2} + \dots + l_1l_n = 0$, it follows from (28) and (12) that the image of the homomorphism

$$\varphi\colon \mathbf{G}_{\mathrm{m}} \to D_n, \qquad t \mapsto (t^{-q}x_1, t^{l_1}x_2, \dots, t^{l_1}x_s, t^{l_1 d}x_{s+1}, t^{l_1}x_{s+2}, \dots, t^{l_1}x_n), \quad (34)$$

lies in G. On the other hand, since the numbers $-q$, l_1 and d are positive, it follows from (4) that for every point $a \in \mathbf{A}^n$ the limit $\lim_{t\to 0} \varphi(t) \cdot a$ exists and is equal to $(0, \dots, 0)$. □

We now consider the case when the n-tuple $l_1, \dots, l_n$ contains 0 and ± 1, at least two of the l_i are non-zero, and all of them have the same sign. By (12) there is no loss of generality in assuming that this sign is positive. Conjugating the group G by an element of $N_{\mathrm{GL}_n}(D_n)$, we may assume that

$$\begin{gathered} l_1 = \dots = l_p = 1, \quad l_{p+1} \geqslant 2, \ \dots, \ l_q \geqslant 2, \quad l_{q+1} = \dots = l_n = 0, \\ p \geqslant 1, \quad n > q \geqslant p, \quad q \geqslant 2. \end{gathered} \quad (35)$$

Lemma 10. *Suppose that (35) holds. Take a point $a = (a_1, \dots, a_n) \in \mathbf{A}^n$. Then the following assertions hold.*

(i) *If $a \notin \bigcup_{i=1}^n H_i$, then the orbit $G \cdot b$, where $b = (a_1, \dots, a_q, 0, \dots, 0)$, lies in the closure of the orbit $G \cdot a$, is closed, and $\dim G \cdot b = q - 1$.*

(ii) *If $a \in H_i$, then the group G_a*

(a) *is trivial for $1 \leqslant i \leqslant p$ and $a \in \mathcal{O}_i$ (see Lemma 6);*

(b) *is non-trivial and finite for $p + 1 \leqslant i \leqslant q$ and $a \in \mathcal{O}_i$;*

(c) *has positive dimension for $i > q$.*

Proof. It follows from (28), (12) and (35) that the image of the homomorphism

$$\varphi\colon \mathbf{G}_{\mathrm{m}} \to D_n, \quad t \mapsto (x_1, \dots, x_q, tx_{q+1}, \dots, tx_n),$$

lies in G. Hence the point

$$\lim_{t \to 0} \varphi(t) \cdot a = (a_1, \dots, a_q, 0, \dots, 0) = b \quad (36)$$

lies in the closure of $G \cdot a$.

Assume that $a \notin \bigcup_{i=1}^n H_i$. If the orbit $G \cdot b$ is non-closed, then, as in the proof of Lemma 7, there is a homomorphism

$$\psi\colon \mathbf{G}_{\mathrm{m}} \to G, \qquad t \mapsto (t^{d_1}x_1, \dots, t^{d_n}x_n), \quad (37)$$

such that $c:=\lim_{t\to 0}\psi(t)\cdot b\in\overline{G\cdot b}\setminus G\cdot b$. It follows from (36) that $d_1\geqslant 0,\dots,d_q\geqslant 0$, and it follows from (35), (28), (12) that $d_1l_1+\dots+d_ql_q=0$. Since $l_1,\dots,l_q$ are positive, we conclude that $d_1=\dots=d_q=0$. By (36) this means that $\psi(t)\cdot b=b$ for every t and, therefore, $c=b$, a contradiction. Thus the orbit $G\cdot b$ is closed. Since $a_1,\dots,a_q$ are non-zero, it follows from (28), (12), (4), (35) and (36) that an element $(t_1x_1,\dots,t_nx_n)\in D_n$ lies in G_b if and only if $t_1=\dots=t_q=1$. This proves (i).

Arguing as in the proof of Lemma 8, we get (ii). □

We finally consider the case when one of the numbers $l_1,\dots,l_n$ is equal to ±1 (by (12) there is no loss of generality in assuming that it is equal to 1) and the others are equal to 0.

Lemma 11. *Suppose that $l_i=1$ and $l_j=0$ for $j\neq i$. For every $s\in k$ let $H(s)$ be the hyperplane in $\mathbf{A}^n$ given by the equation $x_i+s=0$. Then the following assertions hold.*

(i) *$\bigcup_{j\neq i}H_j$ is the set of all points with non-trivial G-stabilizer (which automatically has positive dimension).*

(ii) *The open subset $H(s)\setminus\bigcup_{j\neq i}H_j$ of $H(s)$ is an $(n-1)$-dimensional G-orbit, and all $(n-1)$-dimensional G-orbits are of this form.*

Proof. Part (i) follows immediately from (4), (28) and (12). Part (ii) follows from (i), the invariance of $H(\alpha)$ and the equality $\dim G=\dim H(\alpha)=n-1$. □

§ 6. The group $N_{\operatorname{Aut}\mathbf{A}^n}(D_n(l_1,\dots,l_n))$

We start by proving several general assertions on the normalizers of actions on arbitrary affine varieties.

Lemma 12. *Let X be an irreducible affine variety and G an algebraic subgroup of $\operatorname{Aut}X$. Then the following properties are equivalent:*

(i) *$N_{\operatorname{Aut}X}(G)$ is an algebraic subgroup of $\operatorname{Aut}X$;*

(ii) *the natural action of $N_{\operatorname{Aut}X}(G)$ on $k[X]$ is locally finite.*

Proof. (i)⇒(ii). This follows from the fact that the natural action on $k[X]$ of any algebraic subgroup of $\operatorname{Aut}X$ is locally finite (see [9], Proposition 1.9).

(ii)⇒(i). Assume that (ii) holds. Then one can find an $N_{\operatorname{Aut}X}(G)$-invariant finite-dimensional k-vector subspace V in $k[X]$ such that V contains a set of generators of the k-algebra $k[X]$. Hence the homomorphism

$$\rho\colon N_{\operatorname{Aut}X}(G)\to\operatorname{GL}(V^*)$$

determined by the action of $N_{\operatorname{Aut}X}(G)$ on V is an embedding. Consider the $N_{\operatorname{Aut}X}(G)$-equivariant map

$$\iota\colon X\to V^*,\qquad \iota(x)(f):=f(x)\quad\forall\,x\in X,\ f\in V.$$

A standard argument (see [9], Proposition 1.12) shows that $\rho|_G$ is a morphism of algebraic groups and ι is a closed embedding. We identify X with $\iota(X)$ by means of ι, and $N_{\operatorname{Aut} X}(G)$ with $\rho(N_{\operatorname{Aut} X}(G))$ by means of ρ. Then X is a closed subvariety of V^*, and $N_{\operatorname{Aut} X}(G)$ and G are subgroups of $\mathrm{GL}(V^*)$. Moreover, G is closed and

$$N_{\operatorname{Aut} X}(G) \subset N_{\mathrm{GL}(V^*)}(G) \cap \mathrm{Tran}_{\mathrm{GL}(V^*)}(X, X), \tag{38}$$

where

$$\mathrm{Tran}_{\mathrm{GL}(V^*)}(X, X) := \{g \in \mathrm{GL}(V^*) \mid g \cdot X \subset X\}. \tag{39}$$

We claim that $g \cdot X = X$ on the right-hand side of (39). Indeed, since X is irreducible and closed in V^* and $g \in \operatorname{Aut} V^*$, we see that $g \cdot X$ is an irreducible closed subset of X whose dimension is equal to that of X. Hence $\mathrm{Tran}_{\mathrm{GL}(V^*)}(X, X)$ (as well as $N_{\mathrm{GL}(V^*)}(G)$) is a subgroup of $\mathrm{GL}(V^*)$ and, therefore, the right-hand side of (38) is a subgroup of $\mathrm{GL}(V^*)$. Its elements normalize G and their restrictions to X are automorphisms of X. Hence they lie in $N_{\operatorname{Aut} X}(G)$. Therefore,

$$N_{\operatorname{Aut} X}(G) = N_{\mathrm{GL}(V^*)}(G) \cap \mathrm{Tran}_{\mathrm{GL}(V^*)}(X, X). \tag{40}$$

Since G is closed in $\mathrm{GL}(V^*)$ and X is closed in V^*, we obtain respectively that $N_{\mathrm{GL}(V^*)}(G)$ and $\mathrm{Tran}_{\mathrm{GL}(V^*)}(X, X)$ are closed in $\mathrm{GL}(V^*)$ (see [9], Proposition 1.7). This and (40) imply that $N_{\operatorname{Aut} X}(G)$ is closed in $\mathrm{GL}(V^*)$. Thus $N_{\operatorname{Aut} X}(G)$ is an algebraic subgroup of $\operatorname{Aut} X$. □

Theorem 5. *Let X be an irreducible affine variety and G a reductive algebraic subgroup of* $\operatorname{Aut} X$ *such that*

$$k[X]^G = k. \tag{41}$$

In either of the following cases, $N_{\operatorname{Aut} X}(G)$ is an algebraic subgroup of $\operatorname{Aut} X$:

(i) *G has a fixed point in X;*

(ii) *G^0 is semisimple.*

Proof. Take $f \in k[X]$. We shall prove that the k-linear span of the orbit $N_{\operatorname{Aut} X}(G) \cdot f$ is finite-dimensional in both cases. Then the theorem will follow from Lemma 12.

Let $\mathcal{M}(G)$ be the set of isomorphism classes of simple algebraic G-modules. Given an algebraic G-module L, we denote its isotypic component of type $\mu \in \mathcal{M}(G)$ by L_μ.

Since G is reductive, we have (see [10], Section 3.13)

$$k[X] = \bigoplus_{\mu \in \mathcal{M}(G)} k[X]_\mu\,. \tag{42}$$

The group $N_{\operatorname{Aut} X}(G)$ permutes the isotypic components of the G-module $k[X]$.

Since $k[X]_\mu$ is a finitely generated $k[X]^G$-module (see [10], Theorem 3.24), it follows from (41) that

$$\dim_k k[X]_\mu < \infty \quad \forall\, \mu. \tag{43}$$

In view of (42) there are $\mu_1, \ldots, \mu_s \in \mathcal{M}(G)$ such that

$$k[X]_{\mu_i} \neq 0 \quad \forall\, i, \qquad f \in k[X]_{\mu_1} \oplus \cdots \oplus k[X]_{\mu_s}. \tag{44}$$

(i) Assume that X contains a G-fixed point a. Since G-invariant regular functions separate closed orbits (see [10], Theorem 4.7), it follows from (41) that there are no other fixed points in X. Hence a is also fixed by $N_{\mathrm{Aut}\,X}(G)$. Therefore the ideal

$$\mathfrak{m}_a := \{f \in k[X] \mid f(a) = 0\}$$

is $N_{\mathrm{Aut}\,X}(G)$-invariant. Hence every member of the decreasing filtration

$$\mathfrak{m}_a \supset \cdots \supset \mathfrak{m}_a^d \supset \mathfrak{m}_a^{d+1} \supset \cdots \tag{45}$$

is $N_{\mathrm{Aut}\,X}(G)$-invariant. This filtration has the following property (see [25], Corollary 10.18):

$$\bigcap_d \mathfrak{m}_a^d = 0. \tag{46}$$

In view of (45) we have a decreasing system of nested vector subspaces $\{k[X]_\mu \cap \mathfrak{m}_a^d \mid d = 1, 2, \dots\}$. Since they are finite-dimensional (see (43)), there is a number d_μ such that $k[X]_\mu \cap \mathfrak{m}_a^d = k[X]_\mu \cap \mathfrak{m}_a^{d+1}$ for all $d \geqslant d_\mu$. Then (46) yields that actually

$$k[X]_\mu \cap \mathfrak{m}_a^d = 0 \quad \forall\, d \geqslant d_\mu. \tag{47}$$

Take $l \in \mathbb{Z},\ l \geqslant \max\{d_{\mu_1}, \dots, d_{\mu_s}\}$. It follows from (47) that

$$k[X]_{\mu_i} \cap \mathfrak{m}_a^l = 0 \quad \forall\, i = 1, \dots, s. \tag{48}$$

Since the natural projection $\pi\colon k[X] \to k[X]/\mathfrak{m}_a^l$ is an epimorphism of G-modules, we have $\pi(k[X]_\mu) = (k[X]/\mathfrak{m}_a^l)_\mu$ for every $\mu \in \mathcal{M}(G)$. Since

$$\dim_k k[X]/\mathfrak{m}_a^l < \infty$$

(see [25], Proposition 11.4), we obtain the finiteness of the set of $\mu \in \mathcal{M}(G)$ such that

$$k[X]_\mu \neq 0, \qquad k[X]_\mu \cap \mathfrak{m}_a^l = 0. \tag{49}$$

Let $\{\lambda_1, \dots, \lambda_t\}$ be this set. Since (49) holds for $\mu = \mu_1, \dots, \mu_s$ (see (44), (48)), we may assume that

$$\lambda_i = \mu_i, \quad i = 1, \dots, s. \tag{50}$$

The group $N_{\mathrm{Aut}\,X}(G)$ permutes the isotypic components of the G-module $k[X]$ and sends $\mathfrak{m}_a^l$ to itself. Hence this group permutes $k[X]_{\lambda_1}, \dots, k[X]_{\lambda_t}$ and we see that $k[X]_{\lambda_1} \oplus \cdots \oplus k[X]_{\lambda_t}$ is an $N_{\mathrm{Aut}\,\mathbf{A}^n}(G)$-invariant subspace of $k[X]$. By (43), (44) and (50), it is finite-dimensional and contains f. Hence the k-linear span of the orbit $N_{\mathrm{Aut}\,X}(G) \cdot f$ is finite-dimensional, as required.

(ii) Assume that G^0 is semisimple. Since the index $[G : G^0]$ is finite, it follows from Weyl's formula for the dimension of a simple G^0-module that, up to isomorphism, there are only finitely many simple algebraic G-modules whose dimension does not exceed a preassigned constant. Thus we obtain the finiteness of the set of $\mu \in \mathcal{M}(G)$ such that

$$k[X]_\mu \neq 0, \qquad \dim_k k[X]_\mu \leqslant \max_i \dim_k k[X]_{\mu_i}.$$

Let $\{\lambda_1, \dots, \lambda_t\}$ be this set. We may assume that (50) holds. Since $N_{\mathrm{Aut}\,X}(G)$ permutes the isotypic components and preserves their dimensions, $k[X]_{\lambda_1} \oplus \cdots \oplus k[X]_{\lambda_t}$ is invariant under $N_{\mathrm{Aut}\,X}(G)$. The proof can now be completed as in case (i). $\square$

Corollary 9. *Suppose that X and G are as in Theorem 5 and $k = \mathbb{C}$. Assume that X is simply connected and smooth and $\chi(X) = 1$. Then $N_{\operatorname{Aut} X}(G)$ is an algebraic subgroup of* $\operatorname{Aut} X$.

Proof. By a corollary of the étale slice theorem (see [26], Pt. III, Corollary 2, [10], Theorem 6.7), it follows from (41) and the smoothness of X that X is a homogeneous vector bundle over the unique closed G-orbit $\mathcal{O}$ in X. Hence $\mathcal{O}$ is simply connected and $\chi(X) = \chi(\mathcal{O})$. Being affine, $\mathcal{O}$ is isomorphic to G/H for some reductive subgroup H (see [10], Theorem 4.17). Since G/H is simply connected and $\chi(G/H) = 1$, it follows that $G = H$ (see [12], Section 5.1). Hence $\mathcal{O}$ is a fixed point. The desired assertion now follows from Theorem 5(i). □

Corollary 10. *Suppose that G is a reductive algebraic subgroup of* $\operatorname{Aut} \mathbf{A}^n$ *and* $k[\mathbf{A}^n]^G = k$. *Then $N_{\operatorname{Aut} \mathbf{A}^n}(G)$ is an algebraic subgroup of* $\operatorname{Aut} \mathbf{A}^n$.

Proof. Since $\operatorname{char} k = 0$, we may assume by Lefschetz' principle ([27], Section 15.1) that $k = \mathbb{C}$. Since $\mathbf{A}^n$ is simply connected and smooth and $\chi(\mathbf{A}^n) = 1$, the desired assertion follows from Corollary 9. □

Remark 6. The following example shows that condition (41) alone does not generally imply that $N_{\operatorname{Aut} X}(G)$ is an algebraic subgroup (for an irreducible affine X and reductive G).

Example 1. Let G be an algebraic torus of dimension $n \geqslant 2$. We take X to be the underlying variety of the algebraic group G. The automorphism group $\operatorname{Aut}_{\text{gr}} G$ is embedded in $\operatorname{Aut} X$ and is isomorphic to $\mathrm{GL}_n(\mathbb{Z})$. The action of G on X by left translations embeds G in $\operatorname{Aut} X$. These two subgroups generate $\operatorname{Aut} X$. More precisely, $\operatorname{Aut} X = \operatorname{Aut}_{\text{gr}} G \ltimes G$. Therefore $N_{\operatorname{Aut} X}(G) = \operatorname{Aut} X$. Let $g \in \operatorname{Aut}_{\text{gr}} G$ be an element of infinite order, and let $f_1, \ldots, f_n \in k[X]$ be a basis of $\mathrm{X}(G)$. Then $g^d \cdot f_i \in \mathrm{X}(G)$ for every $d \in \mathbb{Z}$ and all $i = 1, \ldots, n$, and the set $C_i := \{g^d \cdot f_i \mid i \in \mathbb{Z}\}$ is finite if and only if the stabilizer of f_i with respect to the cyclic group generated by g is non-trivial. Assume that all the sets $C_1, \ldots, C_n$ are finite. Then there is $d \in \mathbb{Z}$, $d \neq 0$, such that $g^d \cdot f_i = f_i$ for every $i = 1, \ldots, n$. Since $f_1, \ldots, f_n$ is a basis of $\mathrm{X}(G)$, this means that the automorphism g^d is trivial, contrary to our assumption that g is an element of infinite order and $d \neq 0$. Hence C_i is infinite for some i. Since different characters are linearly independent over k (see [9], Lemma 8.1), it follows that the k-linear span of C_i (and hence of the orbit $N_{\operatorname{Aut} X}(G) \cdot f_i$) is infinite-dimensional.

The information obtained above will now be used to prove that the groups $N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \ldots, l_n))$ are algebraic.

Theorem 6. *The subgroups $N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \ldots, l_n))$ of* $\operatorname{Aut} \mathbf{A}^n$ *are algebraic for all $l_1, \ldots, l_n$. Moreover, the following assertions hold.*

(i) $N_{\operatorname{Aut} \mathbf{A}^n}(D_n) = N_{\mathrm{GL}_n}(D_n)$.

(ii) *If* $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$, *then*

$$N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \ldots, l_n)) \subseteq N_{\mathrm{GL}_n}(D_n) \tag{51}$$

in any of the following cases:

(a) *$l_1, \ldots, l_n$ are all non-zero and have the same sign*;
(b) *none of the numbers $l_1, \ldots, l_n$ is equal to ± 1*;
(c) *the n-tuple $l_1, \ldots, l_n$ contains 0 and $+1$ or -1, there are at least two non-zero numbers, and the non-zero numbers have the same sign.*

(iii) *If $l_i = 1$ and $l_j = 0$ for $j \neq i$, then the group $N_{\mathrm{Aut}\,\mathbf{A}^n}(D_n(l_1, \ldots, l_n))$ is isomorphic to $N_{\mathrm{GL}_{n-1}}(D_{n-1}) \times \mathrm{Aff}_1$ and consists of all $(g_1, \ldots, g_n) \in \mathrm{Aff}_n$ of the form*

$$g_j = \begin{cases} t_j x_{\sigma(j)} & \text{for} \quad j \neq i, \\ t_j x_j + s & \text{for} \quad j = i, \end{cases} \tag{52}$$

where $t_1, \ldots, t_n, s \in k$ and σ is a permutation of the set $\{1, \ldots, i-1, i+1, \ldots, n\}$.

Proof. If G is a subgroup of $\mathrm{Aut}\,\mathbf{A}^n$ and $g \in N_{\mathrm{Aut}\,\mathbf{A}^n}(G)$, $a \in \mathbf{A}^n$, then

$$\begin{gathered} g(G \cdot a) = G \cdot g(a), \qquad g G_a g^{-1} = G_{g(a)}, \qquad g(\overline{G \cdot a}) = \overline{g(G \cdot a)} \\ \text{and, if } G \text{ is algebraic,} \quad \dim G \cdot a = \dim g(G \cdot a). \end{gathered} \tag{53}$$

Suppose that $G = D_n$. Lemma 5(i) implies that $\bigcup_{i=1}^n H_i$ is the set of points whose G-stabilizer has positive dimension. This set is g-invariant by (53). Since the restriction of g to the variety $\bigcup_{i=1}^n H_i$ is an automorphism of this variety, g permutes the irreducible components $H_1, \ldots, H_n$. In other words, there is a permutation $\sigma \in S_n$ such that

$$g(H_i) = H_{\sigma(i)} \quad \forall\, i = 1, \ldots, n. \tag{54}$$

Since the ideal in $k[\mathbf{A}^n]$ determined by H_i is generated by x_i, this shows that the polynomial $g^*(x_i)$ divides $x_{\sigma(i)}$, whence $g^*(x_i) = t_i x_{\sigma(i)}$ for some non-zero $t_i \in k$. Therefore $g \in N_{\mathrm{GL}_n}(D_n)$ (see (3), (8)). This proves (i).

We now suppose that $G = D_n(l_1, \ldots, l_n)$, $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$.

Assume that condition (a) holds. Then (53) and Corollary 8 imply that g permutes the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ (see Lemma 6). In other words, there is a permutation $\sigma \in S_n$ such that $g(\mathcal{O}_i) = \mathcal{O}_{\sigma(i)}$ for every i. By Lemma 6 and (53) it follows that g possesses property (54), whence (as shown above) $g \in N_{\mathrm{GL}_n}(D_n)$. This proves (51) in the case when condition (a) holds.

Suppose that condition (b) holds. Then the g-invariance of $\bigcup_{i=1}^n H_i$ follows from (53) and Lemma 8. Now the same argument as for $G = D_n$ completes the proof of (51) in the case when condition (b) holds.

Suppose that condition (c) holds. To prove (51), we may replace G by the group conjugate to G by an appropriate element of $N_{\mathrm{GL}_n}(D_n)$ and assume that (35) holds. The set $\{a \in \mathbf{A}^n \mid \dim G_a > 0\}$ is closed (see [10], § 1.4). It is g-invariant by (53), and Lemmas 5, 10(ii) imply that its $(n-1)$-dimensional irreducible components are $H_{q+1}, \ldots, H_n$. Hence g permutes $H_{q+1}, \ldots, H_n$. Furthermore, it follows from Lemmas 5, 10(ii) that $\mathcal{O}_{p+1}, \ldots, \mathcal{O}_q$ are all G-orbits $\mathcal{O}$ in $\mathbf{A}^n$ with the property that the stabilizer G_a is finite and non-trivial for $a \in \mathcal{O}$. Then we see from (53) that g permutes $\mathcal{O}_{p+1}, \ldots, \mathcal{O}_q$ and, therefore, permutes their closures $H_{p+1}, \ldots, H_q$. Finally, in view of Lemmas 5, 10(ii), all G-orbits $\mathcal{O}$ in $\mathbf{A}^n$ such that G_a is trivial for $a \in \mathcal{O}$ are exhausted by the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_p$ and $G \cdot a$, where $a \notin \bigcup_{i=1}^n H_i$. Since G is reductive, the closure of every G-orbit in $\mathbf{A}^n$ contains a unique closed G-orbit (see [10], Corollary on Russian p. 189). By Lemma 6, for each of the orbits $\mathcal{O}_1, \dots, \mathcal{O}_p$ this closed orbit is the fixed point $(0, \dots, 0)$. On the other hand, by Lemma 10(i), the closed orbit lying in $\overline{G \cdot a}$ for $a \notin \bigcup_{i=1}^n H_i$ has dimension $q - 1 \geqslant 1$ and, therefore, is not a fixed point. Hence g permutes $\mathcal{O}_1, \dots, \mathcal{O}_p$ and, therefore, permutes their closures $H_1, \dots, H_p$. This proves that (54) holds for a certain permutation $\sigma \in S_n$. As above, this enables us to conclude that (51) holds in case (c). This proves part (ii).

We now assume that $l_i = 1$ and $l_j = 0$ for $j \neq i$. It follows from (53) and Lemma 11(i) that the closed set $\bigcup_{j \neq i} H_j$ is invariant under g. Hence g permutes its irreducible components $H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_n$. Then we conclude as above that for $j \neq i$ we have $g_j = t_j x_{\sigma(j)}$ for some $t_j \in k$ and some permutation σ of the set $\{1, \dots, i-1, i+1, \dots, n\}$. Furthermore, (53) and Lemma 11 yield that $g(\mathcal{O}_i)$ is an orbit which is open in some hyperplane $H(c)$ (see the notation in Lemma 11). Since $\overline{\mathcal{O}_i} = H_i$, and x_i and $x_i + c$ are, respectively, the generators of the ideals of the hyperplanes H_i and $H(c)$, we conclude that $g_i = g^*(x_i)$ differs from $x_i + c$ only by a non-zero constant factor: $g_i = t_i x_i + s$ for some $t_i, s \in k$, $t_i \neq 0$. Thus g is of the form (52). Conversely, it is easy to see that every element $g \in \operatorname{Aut} \mathbf{A}^n$ of the form (52) normalizes G. This proves (iii).

Finally, we claim that $N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ is an algebraic subgroup of $\operatorname{Aut} \mathbf{A}^n$ for any numbers $l_1, \dots, l_n$. Indeed, parts (i)–(iii) show that this is the case if either $(l_1, \dots, l_n) = (0, \dots, 0)$, or $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and any of conditions (a), (b), (c) of part (ii) or the condition of part (iii) holds. The only case not covered by these conditions occurs when the n-tuple $l_1, \dots, l_n$ contains numbers of different signs. But then Lemma 9 yields that there are no non-constant $D_n(l_1, \dots, l_n)$-invariant regular functions on $\mathbf{A}^n$. Hence we conclude by Corollary 10 that the group $N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ is algebraic. □

§ 7. Fusion theorems for tori in Aut $\mathbf{A}^n$ and Aut$^*\mathbf{A}^n$

Fusion theorems describe the subgroups that control the fusion of subsets under conjugation. Namely, let G be an abstract group, and let H be a subgroup of G. We say that $N_G(H)$ controls the fusion of subsets of H under conjugation by elements of G if the following property holds.

(F) For any subset $S \subseteq H$ and any element $g \in G$ with $gSg^{-1} \subseteq H$ there is an element $w \in N_G(H)$ such that $gsg^{-1} = wsw^{-1}$ for all $s \in S$.

If the pair (G, H) possesses property (F), we say that the *fusion theorem* holds for H in G. Note that the action of $N_G(H)$ on H by conjugation boils down to the action of the 'Weyl group' $N_G(H)/Z_G(H)$.

Example 2. The fusion theorem holds for H in G in the following cases.

1) G is a finite group and H is an Abelian Sylow p-subgroup of G. (This is a classical result of Burnside.)

2) G is an affine algebraic group and H is a maximal torus in G. (This is a classical result of the theory of algebraic groups; see, for example, [18], Section 1.1.1.)

3) $G = \mathrm{Cr}_n$ and $H = D_n$. (This is a result of Serre [7], Theorem 1.1.) Since every n-dimensional torus in Cr_n is maximal and conjugate to D_n (see Theorem 2 (i), (ii)), one can replace D_n by any n-dimensional torus.

We also recall that there are $(n-3)$-dimensional maximal tori in Cr_n for $n \geqslant 5$ (see Corollary 7(iii)).

Question 1. Does the fusion theorem hold when D_n is replaced by such a torus?

We now prove that the fusion theorem holds for all n-dimensional tori in $\mathrm{Aut}\,\mathbf{A}^n$ and $(n-1)$-dimensional tori in $\mathrm{Aut}^*\mathbf{A}^n$.

Lemma 13. *For every element $g = (g_1, \ldots, g_n) \in \mathrm{Aut}\,\mathbf{A}^n$ there is an element $g' \in \mathrm{SL}_n$ such that if $s, gsg^{-1} \in \mathrm{GL}_n$, then*

$$gsg^{-1} = g's(g')^{-1}. \tag{55}$$

Proof. Write $g_i = g_i^{(0)} + g_i^{(1)} + \cdots$, where $g_i^{(s)}$ is a form of degree s in $x_1, \ldots, x_n$. Since $\mathrm{Jac}(g) \in k$, we have $\mathrm{Jac}(g) = \det(\partial g_i/\partial x_j|_{x_1=\cdots=x_n=0})$. But the right-hand side of this equality is equal to $\det(\partial g_i^{(1)}/\partial x_j)$. Hence,

$$g^{(1)} := \left(g_1^{(1)}, \ldots, g_n^{(1)}\right) \in \mathrm{GL}_n\,. \tag{56}$$

The automorphism $g^{(1)}$ is the differential of the automorphism g at the point $(0, \ldots, 0)$. It follows from (56) that

(a) $g = g^{(1)}$ if $g \in \mathrm{GL}_n$;

(b) $(ga)^{(1)} = g^{(1)}a$ and $(ag)^{(1)} = ag^{(1)}$ if $a \in \mathrm{GL}_n$.

Let s and $t := gsg^{-1}$ be elements of GL_n. By (56) we have $g^{(1)} \in \mathrm{GL}_n$. Since $gs = tg$, it follows from (a), (b) that $g^{(1)}s = tg^{(1)}$. Therefore the product of $g^{(1)}$ by a constant $\alpha \in k$ with $\alpha^n \det g^{(1)} = 1$ can be taken as g'. □

Theorem 7. *The following pairs (G, H) possess property* (F):

(i) (*fusion theorem for n-dimensional tori in* $\mathrm{Aut}\,\mathbf{A}^n$)

$$(G, H) = (\mathrm{Aut}\,\mathbf{A}^n,\ \textit{an } n\textit{-dimensional torus});$$

(ii) (*fusion theorem for $(n-1)$-dimensional tori in* $\mathrm{Aut}^*\mathbf{A}^n$)

$$(G, H) = (\mathrm{Aut}^*\mathbf{A}^n,\ \textit{an } (n-1)\textit{-dimensional torus}).$$

Proof. (i) Suppose that $G = \mathrm{Aut}\,\mathbf{A}^n$, H is a torus and $\dim H = n$. By Theorem 3 we may assume that H is a maximal torus in GL_n. We preserve the notation in the statement of property (F). By Lemma 13 there is an element $g' \in \mathrm{GL}_n$ such that equality (55) holds for every $s \in S$. Therefore $g'S(g')^{-1} = gSg^{-1} \subseteq H$. Since $g' \in \mathrm{GL}_n$, this shows that

$$H' := (g')^{-1}Hg' \tag{57}$$

is another maximal torus in GL_n containing S. The tori H and H' lie in the closed ([9], Ch. I, Section 1.7) subgroup $Z_{\mathrm{GL}_n}(S)$ of the group GL_n and, therefore, are maximal tori in this subgroup. Since maximal tori in any affine algebraic group are conjugate ([9], Ch. IV, Section 11.3), there is an element

$$z \in Z_{\mathrm{GL}_n}(S) \tag{58}$$

such that

$$H' = zHz^{-1}. \tag{59}$$

It follows from (57) and (59) that $w := g'z \in N_{\mathrm{GL}_n}(H)$. It follows also from (58) and (55) that $gsg^{-1} = wsw^{-1}$ for every $s \in S$.

(ii) The same argument applies in the case when $G = \operatorname{Aut}^* \mathbf{A}^n$, H is a torus and $\dim H = n-1$: we only replace GL_n by SL_n, use Theorem 4 instead of Theorem 3 and note that the element g' may be chosen to lie in SL_n by Lemma 13. □

§ 8. Applications: classification of classes of conjugate subgroups

The results obtained above are used in this section to derive the classifications mentioned in the introduction. In particular, we prove a generalization to disconnected groups of Białynicki-Birula's theorems [4], [5] on the linearization of actions on $\mathbf{A}^n$ of tori of dimension $\geqslant n-1$.

Theorem 8. *Let G be an algebraic subgroup of* $\operatorname{Aut} \mathbf{A}^n$ *such that G^0 is a torus. Then the following assertions hold.*

(i) *If* $\dim G = n$ *or* $\dim G = n-1$, *then there is an element* $g \in \operatorname{Aut} \mathbf{A}^n$ *such that* $gGg^{-1} \subset \mathrm{GL}_n$ *and* $gG^0g^{-1} \subset D_n$.

(ii) *If* $G \subset \operatorname{Aut}^* \mathbf{A}^n$ *and* $\dim G = n-1$, *then there is an element* $g \in \operatorname{Aut}^* \mathbf{A}^n$ *such that* $gGg^{-1} \subset \mathrm{SL}_n$ *and* $gG^0g^{-1} = D_n^*$ *(see* (9)*).*

Proof. (i) Since all maximal tori in GL_n are conjugate, it suffices to prove the existence of $g \in \operatorname{Aut} \mathbf{A}^n$ such that $gGg^{-1} \subset \mathrm{GL}_n$.

The group G is reductive. Therefore, by a corollary of the étale slice theorem ([26], Corollary 2, p. 98), the desired assertion holds when $k[\mathbf{A}^n]^G = k$ (here we only use the reductivity of G, and not the stronger condition that G^0 is a torus). In what follows we may thus assume that $k[\mathbf{A}^n]^G \neq k$. This is equivalent to the condition

$$k[\mathbf{A}^n]^{G^0} \neq k. \tag{60}$$

Replacing G by a conjugate group, we may assume by Theorem 3 and (13) that

$$G^0 = D_n(l_1, \dots, l_n) \subset \mathrm{GL}_n\,. \tag{61}$$

The existence of g will be proved if we show that (61) and (60) imply the inclusion

$$G \subset \mathrm{GL}_n\,. \tag{62}$$

Let F be a finite subgroup of G that intersects every connected component of G. Such a subgroup exists (see [20], Lemme 5.11). Then

$$G = FG^0. \tag{63}$$

Since G^0 is a normal subgroup of G, we have $F \subset N_{\operatorname{Aut} \mathbf{A}^n}(D_n(l_1, \dots, l_n))$. By Theorem 6 we obtain that

$$F \subset N_{\mathrm{GL}_n}(D_n) \subset \mathrm{GL}_n \tag{64}$$

if either $(l_1, \ldots, l_n) = (0, \ldots, 0)$, or $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$ and any of conditions (a)–(c) in part (ii) of Theorem 6 holds. By (61) and (63), this proves (62) for these $(l_1, \ldots, l_n)$. Now Lemma 9 and (60) yield that it remains to consider only one possibility for $(l_1, \ldots, l_n)$: the case when $l_i = 1$ and $l_j = 0$ for $j \neq i$ (with some fixed i). We consider this case and take any element $g \in F$. By Theorem 6(iii) we have equalities (52). Since F is finite, the order of g is finite. It follows that $s = 0$ in (52). This and (8) mean that embeddings (64) (and hence (62)) hold in the case under consideration.

(ii) The proof is the same as in (i). We only replace $\operatorname{Aut} \mathbf{A}^n$ by $\operatorname{Aut}^* \mathbf{A}^n$, GL_n by $\mathrm{SL}_n = \mathrm{GL}_n \cap \operatorname{Aut}^* \mathbf{A}^n$, and $D_n(l_1, \ldots, l_n)$ by D_n^*, use Theorem 4 instead of Theorem 3 and take into account that D_n^* is a maximal torus in SL_n. □

Theorem 9 (classification of n-dimensional diagonalizable subgroups of $\operatorname{Aut} \mathbf{A}^n$ up to conjugacy in $\operatorname{Aut} \mathbf{A}^n$). *Up to conjugacy in* $\operatorname{Aut} \mathbf{A}^n$, *the torus* D_n *is the unique* n*-dimensional diagonalizable subgroup of* $\operatorname{Aut} \mathbf{A}^n$.

Proof. By Theorem 8, this follows from the fact that every diagonalizable subgroup of GL_n is conjugate to a subgroup of the torus D_n (see [9], Ch. 3, § 8.2, Proposition, (d)). □

Theorem 10 (classification of $(n-1)$-dimensional diagonalizable subgroups of $\operatorname{Aut}^* \mathbf{A}^n$ up to conjugacy in $\operatorname{Aut}^* \mathbf{A}^n$). *Up to conjugacy in* $\operatorname{Aut}^* \mathbf{A}^n$, *the torus* D_n^* *is the unique* $(n-1)$*-dimensional diagonalizable subgroup of* $\operatorname{Aut}^* \mathbf{A}^n$.

Proof. By Theorem 8, this follows from the fact that every diagonalizable subgroup of SL_n is conjugate to a subgroup of the torus D_n^* (this follows easily from [9], Ch. 3, § 8.2, Proposition, (d)). □

Theorem 11 (classification up to conjugacy in $\operatorname{Aut} \mathbf{A}^n$ of maximal n-dimensional algebraic subgroups G of $\operatorname{Aut} \mathbf{A}^n$ such that G^0 is a torus). *Up to conjugacy in* $\operatorname{Aut} \mathbf{A}^n$, *the subgroup* $N_{\mathrm{GL}_n}(D_n)$ (*see* (8)) *is the unique maximal algebraic subgroup of* $\operatorname{Aut} \mathbf{A}^n$ *whose connected component of the identity is an* n*-dimensional torus.*

Proof. By Theorem 8, this follows from the fact that $D_n = N_{\mathrm{GL}_n}(D_n)^0$. □

Theorem 12 (classification up to conjugacy in $\operatorname{Aut}^* \mathbf{A}^n$ of maximal $(n-1)$-dimensional algebraic subgroups G of $\operatorname{Aut}^* \mathbf{A}^n$ such that G^0 is a torus). *Up to conjugacy in* $\operatorname{Aut}^* \mathbf{A}^n$, *the subgroup*

$$N_{\mathrm{SL}_n}(D_n^*) = N_{\mathrm{GL}_n}(D_n) \cap \mathrm{SL}_n$$

is the unique maximal algebraic subgroup of $\operatorname{Aut}^* \mathbf{A}^n$ *whose connected component of the identity is an* $(n-1)$*-dimensional torus.*

Proof. By Theorem 8, this follows from the fact that $D_n^* = N_{\mathrm{SL}_n}(D_n^*)^0$. □

We write $\mathcal{L}_n$ for the additive monoid of all $(l_1, \ldots, l_n) \in \mathbb{Z}^n$ such that

(a) $l_1 \leqslant \cdots \leqslant l_n$,

(b) $(l_1, \ldots, l_n) \leqslant (-l_n, \ldots, -l_1)$ with respect to the lexicographic ordering on $\mathbb{Z}^n$.

Theorem 13 (classification of $(n-1)$-dimensional diagonalizable subgroups of $\mathrm{Aut}\,\mathbf{A}^n$ up to conjugacy in $\mathrm{Aut}\,\mathbf{A}^n$). (i) *Every $(n-1)$-dimensional diagonalizable subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$ *is conjugate in* $\mathrm{Aut}\,\mathbf{A}^n$ *to a unique subgroup of the form*

$$D_n(l_1, \ldots, l_n), \quad (l_1, \ldots, l_n) \in \mathcal{L}_n \setminus \{(0, \ldots, 0)\}. \tag{65}$$

(ii) *Every subgroup* (65) *is an $(n-1)$-dimensional diagonalizable group.*

Proof. Let G be an $(n-1)$-dimensional diagonalizable subgroup of $\mathrm{Aut}\,\mathbf{A}^n$. Using Theorem 8(i) and the conjugacy in GL_n of every diagonalizable subgroup of this group to a subgroup of D_n and also Corollary 2(ii), we conclude that G is conjugate in $\mathrm{Aut}\,\mathbf{A}^n$ to a subgroup of the form $D_n(l_1, \ldots, l_n)$, $(l_1, \ldots, l_n) \neq (0, \ldots, 0)$. By the definition of $\mathcal{L}_n$ and Corollary 2(iv), we may assume that $(l_1, \ldots, l_n) \in \mathcal{L}_n$. Suppose that G is also conjugate in $\mathrm{Aut}\,\mathbf{A}^n$ to a subgroup $D_n(l'_1, \ldots, l'_n)$ with $(l'_1, \ldots, l'_n) \in \mathcal{L}_n$. Then $D_n(l_1, \ldots, l_n)$ and $D_n(l'_1, \ldots, l'_n)$ are conjugate in $\mathrm{Aut}\,\mathbf{A}^n$, whence Theorem 7(i) yields that they are conjugate in $N_{\mathrm{GL}_n}(D_n)$. It follows by Corollary 2(iv) and the definition of $\mathcal{L}_n$ that $(l_1, \ldots, l_n) = (l'_1, \ldots, l'_n)$. This proves (i).

Part (ii) follows from Corollary 2(ii). □

We deduce the following theorem from Theorems 6, 9 and 13.

Theorem 14. *If G is a diagonalizable subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$ *of dimension $\geqslant n-1$, then $N_{\mathrm{Aut}\,\mathbf{A}^n}(G)$ is an algebraic subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$.

Remark 7. It is easy to see that $n-1$ in Theorem 14 cannot be replaced by a smaller integer.

Theorem 15 (classification of $(n-1)$-dimensional diagonalizable subgroups of $\mathrm{Aut}\,\mathbf{A}^n$ up to conjugacy in Cr_n). (i) *Two diagonalizable $(n-1)$-dimensional subgroups of* $\mathrm{Aut}\,\mathbf{A}^n$ *are conjugate in* Cr_n *if and only if they are isomorphic.*

(ii) *Every $(n-1)$-dimensional diagonalizable subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$ *is conjugate in* Cr_n *to a unique closed subgroup of D_n of the form* $\ker \varepsilon_n^d$, $d \in \mathbb{Z}$.

Proof. This follows from Theorem 13(i) and Corollary 2(i). □

For $n \leqslant 3$, Theorems 2(i), 9, 13 yield a classification of all tori in $\mathrm{Aut}\,\mathbf{A}^n$ up to conjugacy in $\mathrm{Aut}\,\mathbf{A}^n$ except for the one-dimensional tori in $\mathrm{Aut}\,\mathbf{A}^3$, whose classification is given in the following theorem.

Theorem 16 (classification of one-dimensional tori in $\mathrm{Aut}\,\mathbf{A}^3$ up to conjugacy in $\mathrm{Aut}\,\mathbf{A}^3$). *Every one-dimensional torus in* $\mathrm{Aut}\,\mathbf{A}^3$ *is conjugate in* $\mathrm{Aut}\,\mathbf{A}^3$ *to a unique torus of the form $T(l_1, l_2, l_3)$, where $(l_1, l_2, l_3) \in \mathcal{L}_3$.*

Proof. Let G be a one-dimensional torus in $\mathrm{Aut}\,\mathbf{A}^3$. By Theorem 3, it is conjugate in $\mathrm{Aut}\,\mathbf{A}^3$ to some torus $T(l_1, l_2, l_3)$. Using (8) and the equality $T(l_1, l_2, l_3) = T(-l_1, -l_2, -l_3)$, we may assume that $(l_1, l_2, l_3) \in \mathcal{L}_3$. Suppose that G is also conjugate in $\mathrm{Aut}\,\mathbf{A}^3$ to a torus $T(l'_1, l'_2, l'_3)$ with $(l'_1, l'_2, l'_3) \in \mathcal{L}_3$. Then $T(l_1, l_2, l_3)$ and $T(l'_1, l'_2, l'_3)$ are conjugate in $\mathrm{Aut}\,\mathbf{A}^3$, whence Theorem 7(i) yields that they are conjugate in $N_{\mathrm{GL}_3}(D_3)$. Using Corollary 2(iv), the definition of $\mathcal{L}_n$ and Lemma 4, we now easily deduce that $(l_1, l_2, l_3) = (l'_1, l'_2, l'_n)$. □

§ 9. Jordan decomposition in $\mathbf{Cr}_n$. Torsion primes of the Cremona group

Although the Cremona groups are infinite-dimensional (this has a precise meaning; see [13]), the analogies between them and algebraic groups catch the eye: they have the Zariski topology, algebraic subgroups, tori, roots, Weyl groups, ... Serre ([7], Section 1.2) writes about the analogy

"groupe de Cremona de rang $n \longleftrightarrow$ groupe semi-simple de rang n".

Here we briefly touch upon two topics demonstrating that these analogies extend further. The second topic is intimately related to tori in the Cremona groups.

9.1. Jordan decomposition in $\mathbf{Cr}_n$. Let X be an algebraic variety. 'Algebraic families' endow $\operatorname{Bir} X$ with the Zariski topology ([13], [14], Section 2, [7], Section 1.6): a subset of $\operatorname{Bir} X$ is closed if and only if its inverse image under every algebraic family $S \to \operatorname{Bir} X$ is closed. For every algebraic subgroup G in $\operatorname{Bir} X$ and any subset Z of G, the closure of Z in this topology coincides with its closure in the Zariski topology of G. In particular, G is closed in $\operatorname{Bir} X$.

An element $g \in \operatorname{Bir} X$ is said to be *algebraic* if there is an algebraic subgroup G of $\operatorname{Bir} X$ containing g. This is equivalent to saying that the closure in $\operatorname{Bir} X$ of the cyclic subgroup generated by g is an algebraic subgroup. If G is affine, then we have the Jordan decomposition of g in G (see [9], Pt. I, Section 4):

$$g = g_s g_n. \tag{66}$$

We claim that g_s and g_n depend only on g, not on the choice of G. Indeed, let G' be another affine algebraic subgroup of $\operatorname{Bir} X$ containing g and let $g = g'_s g'_n$ be the Jordan decomposition in G'. Since $G \cap G'$ is a closed subgroup of G and G', there is a Jordan decomposition $g = g''_s g''_n$ in $G \cap G'$. Applying the theorem on the preservation of the Jordan decomposition under homomorphisms (see [9], Theorem 4.4(4)) to the embeddings $G \hookleftarrow G \cap G' \hookrightarrow G'$, we see that $g_s = g''_s$, $g_n = g''_n$, $g'_s = g''_s$, $g'_n = g''_n$. Hence if we call (66) the *Jordan decomposition in* $\operatorname{Bir} X$, we get a well-defined notion.

By [28], every algebraic subgroup of Cr_n is affine. Therefore every algebraic element of Cr_n admits a Jordan decomposition.

The Jordan decompositions in algebraic groups have a number of well-known properties ([29], [9]). Here are some examples.

(a) Every semisimple element of a connected group lies in a torus of this group.

(b) The set of all unipotent elements is closed.

(c) The conjugacy class of every semisimple element of a connected reductive group is closed.

(d) The closure of the conjugacy class of every element g in a connected reductive group contains g_s.

We recall that Cr_n is connected and, for $n = 1, 2$, simple [14]. Moreover, the groups $\operatorname{Aut} \mathbf{A}^n$, $\operatorname{Aut}^* \mathbf{A}^n$ are connected and $\operatorname{Aut}^* \mathbf{A}^n$ is simple [30], [31]. We naturally encounter the following question.

Question 2. Are there analogues or modifications of the properties listed above for the groups Cr_n, $\mathrm{Aut}\,\mathbf{A}^n$ and $\mathrm{Aut}^*\mathbf{A}^n$?

For example, property (a) holds for Cr_n when $n = 1$, but fails when $n > 1$. Indeed, if $n > 2$, then by Theorem 4.3 in [15] the group Cr_n contains a semisimple element of order two which does not lie in any connected algebraic subgroup of Cr_n. If $n = 2$, then our Theorem 2(ii) and Corollary 5 yield that the elements of order $d < \infty$ in Cr_n which can be included in tori constitute a single conjugacy class while, for example, for even d the set of all elements of order d is a union of infinitely many conjugacy classes [32]. On the other hand, property (a) holds for the groups $\mathrm{Aut}\,\mathbf{A}^2$ and $\mathrm{Aut}^*\,\mathbf{A}^2$ because the action of any finite group on $\mathbf{A}^2$ is linearizable [33].

9.2. Torsion primes of the Cremona group. Let G be a connected reductive algebraic group and let p be a prime. We recall (see [18], Section 1.3 and the references therein) that p is called a *torsion prime of the group* G if there is a finite Abelian p-subgroup of G not contained in any torus of G. The set $\mathrm{Tors}(G)$ of all torsion primes of G admits various interpretations. For example, $p \in \mathrm{Tors}(G)$ for $k = \mathbb{C}$ if and only if $\bigoplus_i \mathrm{H}_i(G, \mathbb{Z})$ contains an element of order p (this explains the terminology). The task of calculating $\mathrm{Tors}(G)$ reduces to the case when G is simply connected. For every simple group G, the set $\mathrm{Tors}(G)$ is explicitly described.

Since we can speak of tori in groups like $\mathrm{Bir}\,X$, where X is an algebraic variety, we get a well-defined notion by replacing G in the definition above by (a subgroup of) $\mathrm{Bir}\,X$. In particular, we can replace G by Cr_n, $\mathrm{Aut}\,\mathbf{A}^n$ or $\mathrm{Aut}^*\mathbf{A}^n$. This gives us a definition of the torsion primes of these groups. We denote the corresponding sets of torsion primes by $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n)$ and $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$ respectively. It is natural to ask the following question (concerning $\mathrm{Tors}(\mathrm{Cr}_n)$, it was posed and discussed in [34]).

Question 3. What are the sets $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n)$ and $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$?

Since $\mathrm{Cr}_1 = \mathrm{PGL}_2$, we have

$$\mathrm{Tors}(\mathrm{Cr}_1) = \{2\}. \tag{67}$$

By [32], for $d = 2, 3, 5$ there are infinitely many conjugacy classes of elements of order d in the group Cr_2. On the other hand, as already stated at the end of § 9.1, the elements of order d in Cr_2 that are contained in tori form only one conjugacy class. Hence 2, 3 and 5 are torsion primes of the group Cr_2. Consider a prime $p > 5$. By Theorem E in [35], every element $g \in \mathrm{Cr}_2$ of order p lies in a subgroup isomorphic to $\mathrm{Aut}\,\mathbf{P}^2 = \mathrm{PGL}_3$. Hence g lies in a torus. Finally, according to [36], Theorem B, p. 146, there is a unique (up to conjugacy) non-cyclic finite Abelian p-group in Cr_2. This group is denoted by 0.mn and lies in a maximal torus of a subgroup isomorphic to $(\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1))^0 = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Thus we conclude that the torsion primes of Cr_2 are equal to those of the exceptional simple algebraic group E_8:

$$\mathrm{Tors}(\mathrm{Cr}_2) = \{2, 3, 5\}. \tag{68}$$

Since $\mathrm{Aut}\,\mathbf{A}^1 = \mathrm{Aff}_1$, it follows from [33] that every finite subgroup of $\mathrm{Aut}\,\mathbf{A}^n$ (resp. $\mathrm{Aut}^*\mathbf{A}^n$) for $n \leqslant 2$ is contained in a subgroup isomorphic to GL_n (resp. SL_n).

Therefore,

$$\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n) = \mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n) = \{\varnothing\} \quad \text{for} \quad n \leqslant 2.$$

For $n \geqslant 3$ there is no comprehensive information about the sets $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n)$ and $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$. It follows from Theorem 4.3 in [15] and (67), (68) that

$$\mathrm{Tors}(\mathrm{Cr}_n) = \{2, \dots\} \quad \forall\, n. \tag{69}$$

By [37], there is a 3-elementary Abelian subgroup of rank 3 in Cr_2 (hence this subgroup is isomorphic to $(\boldsymbol{\mu}_d)^3$). Since Cr_1 contains a cyclic subgroup of order 3 and one can embed the direct product of Cr_2 and $n-2$ copies of Cr_1 in Cr_n for $n \geqslant 3$, we see that there is a 3-elementary Abelian subgroup G of rank $n+1$ in Cr_n. But Lemma 2(i) yields that for every prime p, the rank of any elementary Abelian p-subgroup of an r-dimensional torus does not exceed r. Using this and Theorem 2(i), we see that G is not contained in a torus of Cr_n. By (68) and (69), it follows that

$$\mathrm{Tors}(\mathrm{Cr}_n) = \{2, 3, \dots\} \quad \forall\, n \geqslant 2.$$

Question 4. What is the minimal n such that 7 lies in one of the sets $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n)$ and $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$?

Question 5. Are these sets finite?

Bibliography

[1] V. L. Popov, *Problems for problem session*, 2005, arXiv: math/0504517.

[2] V. L. Popov, "Roots of the affine Cremona group", *Affine algebraic geometry* (Seville, Spain 2003), Contemp. Math., vol. 369, Amer. Math. Soc., Providence, RI 2005, pp. 12–13.

[3] A. Liendo, "Roots of the affine Cremona group", *Transform. Groups* **16**:4 (2011), 1137–1142.

[4] A. Białynicki-Birula, "Remarks on the action of an algebraic torus on k^n", *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 177–181.

[5] A. Białynicki-Birula, "Remarks on the action of an algebraic torus on k^n. II", *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 123–125.

[6] J. Blanc, "Conjugacy classes of affine automorphisms of $\mathbf{K}^n$ and linear automorphisms of $\mathbf{P}^n$ in the Cremona groups", *Manuscripta Math.* **119**:2 (2006), 225–241.

[7] J.-P. Serre, "Le groupe de Cremona et ses sous-groupes finis", *Séminaire Bourbaki. Volume* 2008/2009. *Exposés* 997-1011, Astérisque, vol. 332, Soc. Math. de France, Paris 2010, pp. 75–100.

[8] J.-P. Serre, "Faisceaux algébriques cohérents", *Ann. of Math.* (2) **61**:2 (1955), 197–278.

[9] A. Borel, *Linear algebraic groups*, Grad. Texts in Math., vol. 126, Springer-Verlag, New York 1991.

[10] V. L. Popov and E. B. Vinberg, "Invariant theory", *Algebraicheskaya Geometriya* – 4, Itogi Nauki Tekhn. Ser. Sovrem. Probl. Mat. Fundam. Napr., vol. 55, VINITI, Moscow 1989, pp. 137–309; English transl., *Algebraic Geometry* IV, Encyclopaedia Math. Sci., vol. 55, Springer-Verlag, Berlin 1994, pp. 123–284.

[11] G. Laumon, “Comparaison de caracteristiques d'Euler–Poincaré en cohomologie l-adique”, *C. R. Acad. Sci. Paris Ser.* I *Math.* **292** (1981), 209–212.

[12] H. Kraft and V. L. Popov, “Semisimple group actions on the three-dimensional affine space are linear”, *Comment. Math. Helv.* **60**:3 (1985), 466–479.

[13] C. P. Ramanujam, “A note on automorphism groups of algebraic varieties”, *Math. Ann.* **156**:1 (1964), 25–33.

[14] J. Blanc, “Groupes de Cremona, connexité et simplicité”, *Ann. Sci. École Norm. Sup.* (4) **43**:2 (2010), 357–364.

[15] V. L. Popov, “Some subgroups of the Cremona groups”, *Affine algebraic geometry*, Proceedings of the conference, Osaka, Japan, 3–6 March 2011, World Scientific, New Jersey, pp. 213–242.

[16] E. B. Vinberg, *A course in algebra*, Faktorial Press, Moscow 2002; English transl., Grad. Stud. Math., vol. 56, Amer. Math. Soc., Providence, RI 2003.

[17] M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, MA 1964; Russian transl., Nauka, Moscow 1972.

[18] J.-P. Serre, “Sous-groupes finis des groupes de Lie”, *Séminaire Bourbaki. Volume* 1998/99. *Exposés* 850–864, Astérisque, vol. 266, Soc. Math. de France, Paris 2000, pp. 415–430.

[19] I. R. Shafarevich and A. O. Remizov, *Linear algebra and geometry*, Fizmatlit, Moscow 2009; English transl., Springer-Verlag, Heidelberg 2013.

[20] A. Borel and J.-P. Serre, “Théorèmes de finitude en cohomologie galoisienne”, *Comment. Math. Helv.* **39** (1964), 111–164.

[21] M. Demazure, “Sous-groupes algébriques de rang maximum du groupe de Cremona”, *Ann. Sci. École Norm. Sup.* (4) **3** (1970), 507–588.

[22] S. Kaliman, M. Koras, L. Makar-Limanov, and P. Russell, “$\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable”, *Electron. Res. Announc. Amer. Math. Soc.* **3** (1997), 63–71.

[23] N. Gupta, *A counter-example to the cancellation problem for the affine space* $\mathbf{A}^3$ *in characteristic* p, 2012, arXiv: 1208.0483.

[24] V. L. Popov, “Stability criteria for the action of a semisimple group on a factorial manifold”, *Izv. Akad. Nauk SSSR Ser. Mat.* **34**:3 (1970), 523–531; English transl., *Math. USSR-Izv.* **4**:3 (1970), 527–535.

[25] M. F. Atiyah and I. G. McDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA 1969; Russian transl., Mir, Moscow 1972.

[26] D. Luna, “Slices étalés”, *Bull. Soc. Math. France* **33** (1973), 81–105.

[27] J. Harris, *Algebraic geometry*, Grad. Texts in Math., vol. 133, Springer-Verlag, New York 1992; Russian transl., MCCME, Moscow 2006.

[28] H. Matsumura, “On algebraic groups of birational transformations”, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **34** (1963), 151–155.

[29] R. Steinberg, “Regular elements of semisimple algebraic groups”, *Inst. Hautes Études Sci. Publ. Math.* **25**:1 (1965), 49–80.

[30] I. R. Shafarevich, “On some infinite dimensional groups”, *Rend. Mat. e Appl.* (5) **25**:1–2 (1966), 208–212.

[31] I. R. Shafarevich, “On some infinite-dimensional groups. II”, *Izv. Akad. Nauk SSSR Ser. Mat.* **45**:1 (1981), 214–226; English transl., *Math. USSR-Izv.* **18**:1 (1982), 185–194.

[32] J. Blanc, “The number of conjugacy classes of elements of the Cremona group of some given finite order”, *Bull. Soc. Math. France* **135**:3 (2007), 419–434.

[33] T. Igarashi, *Finite subgroups of the automorphism group of the affine plane*, Thesis, Osaka University 1977.

[34] V. L. Popov, *Tori in Cremona groups*, Essential Dimension and Cremona Groups, talk delivered at the International workshop, Chern Institute of Mathematics, Nankai University, Tianjin, China, June 11–15, 2012, http://www.nim.nankai.edu.cn/activites/conferences/hy20120611/index.htm.

[35] T. de Fernex, "On planar Cremona maps of prime order", *Nagoya Math. J.* **174** (2004), 1–28.

[36] J. Blanc, *Finite abelian subgroups of the Cremona group of the plane*, Thèse No. 3777, Université de Genève, Faculté des Sciences 2006.

[37] A. Beauville, "p-elementary subgroups of the Cremona group", *J. Algebra* **314**:2 (2007), 553–564.

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